Lesson 2 : Matter-dominated and radiation-dominated universes

Notes from Prof. Susskind video lectures publicly available on YouTube

Introduction

After a brief review of what we have done so far, we shall move on to generalizations.

We worked out the equations of an expanding universe. They were Newton's equations. Do they give a satisfactory description of the universe ? Yes, they get it right for the most part. Let's explain why.

Einstein's equations have to do with curved spacetime. The universe that we are ultimately going to study will indeed be a curved spacetime. And in fact some versions of it even will have a curved space. That simply means that, leaving time aside, space itself is curved.

Consider to begin with a two-dimensional world – what mathematicians call a two-dimensional variety. If you measure triangles on it, or if you do various kinds of geometric exercices on it, you will discover perhaps that it is curved.

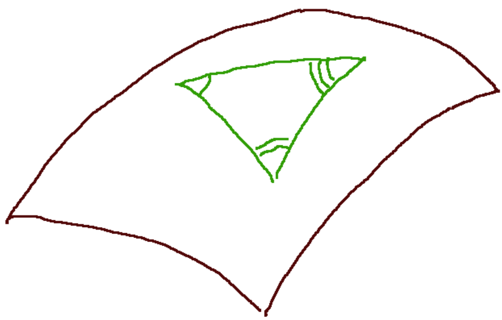


Figure 1 : Positively curved two dimensional world.

If we lived in a two dimensional space, like little bugs in Flatland [1], it would be easy to figure out what we mean by a curved space. We could be living indeed in a flat Euclidean plane – which has no curvature. Or we could, in fact, be living on the surface of a very big sphere which is a two-dimensional variety with positive curvature. Or we could even be living in a fancier shaped space (negatively shaped simply connected, torus-like, etc.).

When embedding the surface of the sphere in three dimensions (figure 1), we see clearly what is meant by curvature. But, as we saw extensively in volume 4 of *The Theoretical Minimum* collection on general relativity, the curvature is intrinsic. It has nothing to do with the embedding space we use for convenience to visualize its shape.

Moreover, the notion of positive or negative curvature, compared to flat Euclidean space, can easily be extended to three dimensional varieties – not so much visually, as with equations. The extension of the metric and subsequent equations is straightforward. All of this will be studied at length in lesson 3.

At present our universe, the spatial part of our spacetime, looks pretty flat. But it is possible that it will turn out, on the average, to be curved. What do we mean by "on average"? Let's go back once again to two dimensions. Any portion of the surface of the Earth not too big, say one thousand kilometers in radius, may have mountains and valleys, so is clearly not flat, but, unless we need precise measures, it can be considered *on average* to be simply a

1. Famous book, written by Edwin Abbott, published in England in 1884, describing creatures living in a two-dimensional world.

flat plane.

Similarly, in the three-dimensional variety that is the spatial universe we live in, if we look only at very neighboring galaxies – meaning galaxies at less than say a billion light years from us – the universe may seem flat, like a 3D Euclidean space. In three dimensions, instead of checking the sum of the angles of a triangle, or its surface area, like in a plane, we may want to check the volume of a tetrahedron. Is it or not one third the area of its base multiplied by its height ?

We consider a billion light years around us because we think that it is much smaller than what we estimate the radius of curvature of the whole universe, if any, to be.

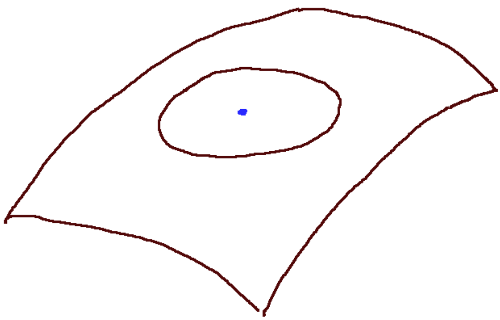


Figure 2 : Locally flat region in a two-dimensional curved world.

If such a region, a billion light years around us, looks flat, it should mean that at least when doing physics only in that portion of the universe we should not have to worry about the fact that the entire universe is curved.

If that is correct, it means that the way galaxies move relative to each other and how they move apart from each other, at least in the region we are considering, can be studied using a 3D Euclidean framework and Newton's equations.

That is what we have been doing. We have been looking at the universe in the small, and studying how a little fraction of it is expanding or not expanding. It is perfectly legitimate, and in fact entirely consistent with Einstein's general relativity, except for one thing.

We would run into trouble if the galaxies, or whatever is present, galaxies, particles, etc., were really moving past each other with a significant fraction of the speed of light. In other words, one of the assumptions is that neighboring things are moving relatively slowly with respect to each other.

Something very far away from us, however, maybe moving with a large velocity relative to us. But as long as the things nearby are moving with non relativistic velocities with respect to us, we can take a small patch – now small could mean several billion light years on the side – and study it without using any relativity.

On the other hand, if we discover that there are particles moving with close to the speed of light past each other then of course we will have to modify the equations. And indeed there are particles moving fast by comparison with the speed of light past us. What are they ? You may think of neutrinos which are particles with very small mass moving close to the speed of light.

Even more simply, think of photons. They are massless particles, with energy, moving at the speed of light. Around us there are photons from the Sun, but not only. There would be photons even if there was no Sun. In the same way that it is filled with galaxies, the universe is also filled with homogeneous radiation. This radiation does move with the speed of light. As a consequence we ought to modify our equations somehow to account for this. That will be one of the tasks of this lesson.

Let's first briefly review, however, the model of evolving Newtonian universe we built in the last lesson, and the most important equations we derived.

Review of the Newtonian universe

We established from observations that the universe is filled with galaxies, which can be viewed as forming some sort of homogeneous gas, with a certain number of particle per cubic volume. In other words, at any given time, the universe has a density of mass constant in space. We call it ρ.

The universe appears to be expanding in a uniform manner – like a cake in the owen, whose raisins would be the galaxies, see figures 3a and 3b. If the universe was contracting the same reasoning would apply.

Figure 3a : Homogeneous distribution of galaxies in the universe, seen as a gas of particles, viewed at time t_1.

Figure 3b : Same region of the universe viewed at time t_2.

Therefore the density of mass ρ changes with time. With an expanding universe it becomes smaller and smaller as time goes on.

Now comes the crux of the model we built. Instead of using a 3D Euclidean reference frame, with a fixed distance between planes of coordinates $X = 1$, $X = 2$, $X = 3$ etc., we introduce an astute *moving grid*, which accompanies the grand movement of galaxies. We can place its origin anywhere, it plays no role at first.

With respect to this fictitious moving grid, the coordinates of the galaxies don't change (except for possible small local movements which we ignore). If galaxy A has coordinates $X = X_1$, $Y = Y_1$ and $Z = Z_1$ at time t_1 in the moving grid, its coordinates at time t_2 in the moving grid will be the same.

Nevertheless, the actual distance between two galaxies A and B, with grid-coordinates respectively $(X_1,\ Y_1,\ Z_1)$ and $(X_2,\ Y_2,\ Z_2)$, *does change* with time. It is not given by direct application of Pythagoras theorem, which would be

$$\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

In the simple model we built, we introduce a scale factor a to take into account the inflation of the universe between times t_1 and t_2. This scale factor depends on time. At a given time t the actual distance between A and B is

$$a(t)\ \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \qquad (1)$$

At another time, the square root part in formula 1 remains the same, but the value of the scale factor can be different.

Notice that this scale factor does not depend on space because the expansion of the universe appears to be the same everywhere. It depends only on time. At a given time, it is the same parameter that we use to measure the distance between any pair of galaxies, be they close to each other or far apart.

You may wonder how big is $a(t)$ at a given time t ? The answer is that it depends on the moving grid we chose. If at time t_0, the moving grid happens to correspond to the standard 3D Euclidean reference frame which we use to measure distances, then at that time we have $a(t_0) = 1$. But if at that time t_0 the moving grid was for instance twice as fine, then the actual distance between $X = 4$ and $X = 5$ at time t_0 would be $1/2$, therefore $a(t_0)$ would be $1/2$.

At this stage the actual value of the scale parameter $a(t)$, at any given time, has no physical meaning. Later on we will discuss more what a means, but for now on a flat page (or rather in a flat 3D space) a doesn't mean anything by itself. The scale parameter a is just some sort of bookkeeping device to label locations – which are moving – in the universe.

On the other hand ratios of values of a may mean something. Let's continue to suppose that the universe is expanding. Galaxies move embedded in the moving grid. We also expressed it by saying that "they are frozen with the grid". The actual physical distance between galaxies is growing. Suppose that, over a period of time, the scale factor $a(t)$ doubled. That has meaning. It means the distance between every pair of galaxies doubles.

Ratios of a at different times are recording a history of how the universe is evolving – expanding or contracting. The fact that a has a *relative* meaning explains why our equations tend to only involve ratios of things with a.

Just like $a(t)$, at a given time t, depends on the scale of the grid we have selected at some initial time, so does the time derivative of $a(t)$, denoted $\dot{a}(t)$. But the ratio $\dot{a}/a$ is independent of the choice of initial scale of the moving grid. More generally measures involving ratios of quantities proportional to a, with a, are independent of the initial scale of the grid and physically meaningful measures.

The most fundamental one is

$$H = \frac{\dot{a}(t)}{a(t)} \tag{2}$$

It is the *Hubble constant.* It is constant in space, but variable in time.

Another important one is the density with respect to the grid. By this we mean the quantity of mass in one cubic grid cell of sides $\Delta X = \Delta Y = \Delta Z = 1$. Such a volume is a box of evolving size. We called that density ν. And we have the simple relation

$$\rho = \frac{\nu}{a^3} \tag{3}$$

Once the grid initial scale has been selected, this grid-density ν remains constant because the galaxies and the grid move together. Therefore the number of galaxies per unit grid cell, or the mass per unit grid cell, stay the same. On the other hand, the standard mass density ρ – that is the average mass per cubic meter in the universe, which is independent of space – depends on time. It decreases if the universe expands.

To finish this quick review of the evolving Newtonian universe, let's turn to the equations that we derived in the last lesson.

We started from the universe uniformly filled with galaxies, which we think of as particles, see figure 4. We selected arbitrarily a fixed point to be the origin of both the fixed 3D Euclidean reference frame and of our moving grid. And we placed the Milky Way, that is ourselves, at that origin O.

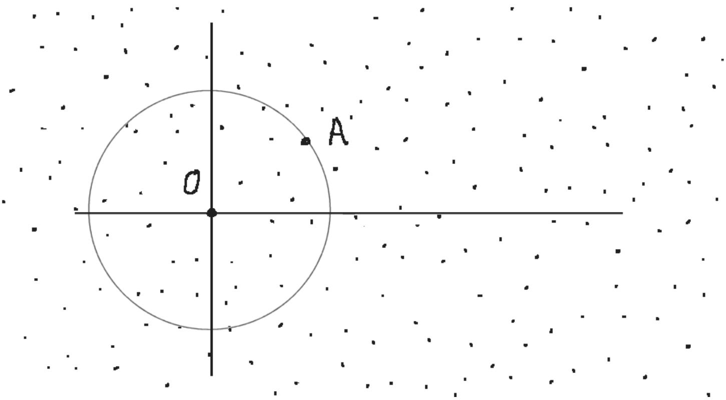


Figure 4 : Universe uniformly filled with galaxies. Origin O and galaxy A.

We looked at a galaxy A which is at some location on the grid. Its grid-coordinates are $(X,\ Y,\ Z)$, which won't change. For simplicity we denote this vector location with just the letter X. Again, it is a constant, since it is the grid-coordinate of A. And we studied Newton's equations for the motion of galaxy A. The calculations are frame dependent. We carry them in the grid with origin O.

As the reader remembers, according to Newton's theorem, the galaxies farther away from O than A globally exert no net force on A. And the galaxies within the sphere in figure 4 have a net effect equivalent to a single point of the same total mass at the origin.

The radius of the sphere is aX. And the mass of all the galaxies within the sphere, which are attracting A, is

$$M = \frac{4\pi}{3} a^3 X^3 \rho \tag{4}$$

The relation between ρ and ν given by equation (3) comes in handy. Equation (4) can be rewritten

$$M = \frac{4\pi}{3} X^3 \nu \tag{5}$$

We could now write the force exerted on A, but let's rather work with energies. In physics calculations it is usually simpler and always more elegant. The total energy of A is the sum of its kinetic energy and its potential energy. The kinetic energy of A is

$$\frac{1}{2} m \dot{a}^2 X^2 \tag{6}$$

And its potential energy of is

$$-\frac{mMG}{aX} \tag{7}$$

We looked at the case where the total energy of A was zero, which corresponds exactly to the escape velocity. This gives the equation

$$\frac{1}{2} m \dot{a}^2 X^2 = \frac{mMG}{aX} \tag{8}$$

Using equation (5) and pushing the algebra we get

$$\frac{1}{2} m \dot{a}^2 X^2 = \frac{4\pi m G}{3aX} X^3 \nu$$

$$\dot{a}^2 X^2 = \frac{8\pi\ G}{3a}\ X^2\ \nu$$

Finally

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\nu}{3}\ \frac{1}{a^3} \tag{9}$$

We stress that this was in the case of zero energy, that is when the kinetic energy and the potential energy of each galaxy have exactly equal magnitude and opposite sign. In other words, every galaxy is exactly at the escape velocity – just in the knife edge between being able to escape and not being able to escape.

In the course of the calculations, we replaced formula (4) involving ρ with formula (5) involving ν. The reason why it is more convenient is that ρ, which is the density per cubic meter, varies with time. Whereas ν, which is the density per cubic box in the moving grid, does not.

Indeed if ν is calculated using only ordinary massive particles sitting in the universe, which are never destroyed, never created, that is protons and such like[2], then ν is constant. Let's forget for the moment that protons do decay. For convenience we postulate that protons and galaxies are forever. We won't believe that forever, but we will believe for the moment that they are forever.

What is the numerical value of ν? As for $a(t)$, there is no absolute answer. It depends on the moving grid we chose. Said more precisely, it depends on the scale of the moving

2. They are called *baryons*.

grid chosen at some initial time.

In any case, on the right-hand side of equation (9) all the stuff in front of $1/a^3$ is a numerical constant. By judiciously choosing the initial scale of the moving grid, or our units, we can, if we like, make $8\pi G\nu/3$ equal to 1. That is how we reached the following nice equation for the evolution of a Newtonian universe. The scale factor satisfies

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^3} \tag{10}$$

We solved it in chapter 1 by trying out a solution of the form $a(t) = ct^p$.

Because it is an important type of equation in cosmology, let's see now how to find the solution without any prior guess.

We take the square root of both sides. First of all we will work with the positive square roots, then we will comment on what would happen if we worked with the other sign on one side.

$$\frac{\dot{a}}{a} = \frac{1}{a^{3/2}}$$

$$\dot{a} = \frac{1}{a^{1/2}}$$

$$\frac{da}{dt} = \frac{1}{\sqrt{a}}$$

Then comes the clever step. Instead of thinking of a as a function of time, we are going to think of time as a function of a. In other words the independent variable t will become the dependent variable. Conversely the dependent variable a will become the independent variable. Since the right-hand side of equation (10) never goes to zero, the left-hand side cannot either, so $\dot{a}$ cannot change sign. Therefore the function $a(t)$ has an inverse function $t(a)$, the derivative of which is simply dt/da. So our equation becomes

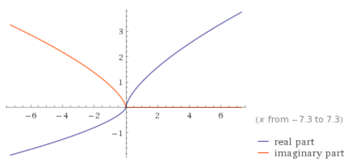
$$\frac{dt}{da} = \sqrt{a}$$

Now it is just coasting. We look for a function of a whose derivative is $\sqrt{a}$. It is

$$t(a) = \frac{2}{3}\ a^{3/2}$$

The constant $2/3$ in front will get absorbed in other constants, so we can ignore it. Finally taking the power $2/3$ on both sides we get

$$a = t^{2/3} \tag{11}$$



source: /www.wolframalpha.com/

Figure 5 : Function $a(t) = t^{2/3}$.

Figure 5 shows the complete graph of the function $a(t) = t^{2/3}$ produced by the site www.wolframalpha.com. But we are only concerned with the real and positive part on the right side of it.

We see that $a(t)$ increases with time, but getting flatter and flatter as time goes on. The flattening reflects the deceleration of the universe. But it never comes to rest. As mentioned, we could see that it never comes to rest because when we look at equation (10) there is no point at which $\dot{a}/a$ becomes 0.

Another way to see that it will decelerate is from the physics. It is equivalent to a particle moving away from a fixed force center. So it will be decelerate because of the attraction by the force center.

If we had taken the other square root solution from equation (10), we would have gotten a contracting universe accelerating up to $a = 0$, which is indeed the other possible motion. What is the logic ?

From the equation we cannot tell whether the universe is expanding or contracting. Both cases are possible from the equation. It is exactly the same as for an object in a force center field created by an attracting object. Think of the Sun and a planet, or of a planet and a rock above its surface.

The equations of Newton will tell you what the acceleration of the rock is. But they will not tell you at an instant t whether the rock is moving away or toward the planet. It depends on the initial conditions. Let's forget that the rock may be in orbit. Let's consider a rock that is moving radially. One thing is sure : the rock will not be standing still. It may be *momentarily* at rest. It might have come up and stop but it is not going to stand still. It is accelerating back toward the planet.

In the same way you cannot tell from the equation whether the universe is expanding or contracting. That has to do with the two possible solutions to equation (10), whether da/dt is positive or negative. If it is contracting it is negative. If it is expanding it is positive. It is left as an exercise for the reader to work out the negative solution and plot it.

The analogy between a rock moving radially with respect to an attractive object and all the galaxies of the universe expanding or contracting is possible because of the fact ultimately coming from observation that all the galaxies move according to a grand coherent movement. They don't zip past each other like the molecules of gas in a vase or a room.

On the average they tend to move as if they were embedded in this grid we introduced. Given that they are embedded in the grid, the only option they have is to move radially relative to each other.

Of course, there are other motions on top of the average motion, called peculiar motions. For instance the Earth moves around the Sun, so they do not illustrate the general expansion. Even Andromeda and the Milky Way have a local peculiar movement which seems inconsistent with the grand expansion of the universe! But if we look a few galaxies away from us, we observe the expansion.

In other words, the real motion of galaxies is the combination of an average flow, as for a river, and local motions. Particles in a river follow the general current, and at the same time there are swirls, eddies and other local deviations from the main flow. How fast is any given molecule moving ? We cannot say. But we do know on the average that clumps of molecules, sufficiently averaged, are moving with whatever the velocity of the river is. On top of that there is the peculiar motion of the molecules, and even floating particles, relative to each other.

That is the way to think about the expansion of the universe : a general expanding flow, with, on top of it, fluctuations. And we are ignoring the fluctuations at the moment.

We are now finished with our review of lesson 1. I redid it because I always think it is worthwhile redoing the most important derivations. The slighty different explanations and calculations may help the reader strengthen his or her understanding of this first simple model of an evolving Newtonian universe.

There are two directions in which we now want to generalize our model.

1) We shall move beyond the assumption that the total energy of each galaxy is zero. In other words, we shall investigate what happens when they are above, or below, escape velocity.

2) We shall study what happens if the universe is made out of other kinds of stuff than just non-relativistic particles moving at relatively slow speed with respect to each other when they are close, radiations in particular ?

Mind you, this will still be story of the universe as it was told around the end of the 20th century. We are not yet reaching the most recent developments of cosmology.

The universe we live in is mostly made of matter. Most of its energy is the masses of particles. It is an $E = mc^2$ kind of energy. And the theory we have developed so far was about that : massive non-relativistic particles.

But we shall also study a fictitious universe only made of photons. Why study that ? Because we are going to discover that, early on in the history of the universe, the most important things were photons. By this we mean that the largest concentration of energy, at an early stage in the universe, was radiation energy.

Let's begin with the first generalisation.

A Newtonian universe not at escape velocity

As usual we will use the principle of conservation of energy. So we go back to the total energy of galaxy A in figure 4. It is the sum of its kinetic energy, given by formula 6, and its potential energy, given by formula 7. In standard notations, we can write it as follows

$$\frac{1}{2}\, m\, V^2 - \frac{mMG}{D} \tag{12}$$

where V is the speed of galaxy A with respect to the origin, m its mass, M the total mass of all the matter in the sphere,

G Newton's constant, and D the distance of the galaxy to O. We want to see what happens when the sum, that is the total energy, expressed by formula (12), is not zero.

What do we know about the total energy of A? We know that it does not change with time, because our model is just the motion of a particle in a fixed background of mass. So let's set the energy to be equal to some constant.

But we have to be careful : what does constant mean? Does it mean a numerical value that is independent of everything else? No, it doesn't. It could depend on which particle we are talking about. In fact if the universe is really homogeneous, the only thing it could depend on is the grid-distance between A and O.

So we will simplify the analysis by taking the galaxy, or particle, A to be at $X = 1$. In other words we are focusing on a very specific particle A. It moves from the origin O but its grid-distance from O is $X = 1$ and doesn't change. We will reintroduce X in the equations in a moment.

Galaxy A has some specific energy and that energy will never change. Let's call it E.

$$\frac{1}{2}\, m\, V^2 - \frac{mMG}{D} = E \tag{13}$$

What is the value of E? We don't know, just like we don't know what is the energy of an object at some distance of the Earth because we have to know how fast it is moving. So we have to study all possible cases.

Now let's work a bit with equation (13). First of all we divide by $m/2$. This yields

$$V^2 - \frac{2MG}{D} = \frac{2E}{m} \tag{14}$$

The right-hand side is still a constant. And since we did not know what E was, we haven't really changed the equation.

On the left-hand side, we have been using the variables V and D because they are usual notations for the two types of energy. But it is time to introduce the scale factor $a(t)$. Of course we have

$$\begin{aligned} D &= aX \\ V &= \dot{a}X \end{aligned} \tag{15}$$

So we can plug that into equation (14). Furthermore we have chosen X, the fixed grid-distance between A and O, to be 1. Therefore we can simply write

$$\dot{a}^2 - \frac{2MG}{a} = C \tag{16}$$

We replaced the constant $2E/m$ on the right-hand side by C. (It is *not* the speed of light.) Equation (16) begins to look similar to equation (9), which we derived in the case of a universe at exactly escape velocity, that is where each galaxy had zero total energy.

To get even closer to equation (9), we can divide both sides of equation (16) by a^2. Remember that the numerical value of the scale factor $a(t)$ doesn't have itself a physical meaning, because it depends on the choice of units for the moving grid. Only ratios of quantities proportional to a or to $\dot{a}$ are meaningful. Dividing by a^2 we get

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{2MG}{a^3} = \frac{C}{a^2} \tag{17}$$

The difference with equation (9) is that the right-hand side is no longer a constant, let alone zero. That is because we are now investigating the case of a Newtonian universe not at escape velocity, and galaxies have a non zero total energy [3].

The next step should by now be familiar. We are going to make the volume of the sphere in figure 4 appear in equation (17), in order to get at a density. If $X = 1$, the actual volume of the sphere in figure 4 is $4\pi a^3/3$. If we denote this volume by $\mathcal{V}$, equation (17) can be rewritten

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi MG}{3\mathcal{V}} = \frac{C}{a^2} \tag{18}$$

$M/\mathcal{V}$ is the actual density of the universe, that is the mass per real volume. We have denoted it ρ. So equation (18) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{C}{a^2} \tag{19}$$

3. Remember that the calculation of the kinetic energy as well as the potential energy is frame dependent. We are reasoning with an origin O placed somewhere arbitrarily, and we are looking at a galaxy at grid-distance 1 from that origin. At another grid-distance from it, the kinetic energy and potential energy of the galaxy in equation (13) would be different.

Finally remember that the actual density ρ, which is variable because the universe expands or contracts, is related to the grid-density ν, which is the fixed quantity of mass per unit grid-volume, by equation (3). If we want to clearly distinguish constants and variables, equation (19) can be rewritten

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\nu}{3}\frac{1}{a^3} + \frac{C}{a^2} \tag{20}$$

By fiddling around we have reached again the Friedmann equation, see equation (9) for the previous instance. But this time the right-hand side contains also a term inversely proportional to a^2.

The term C/a^2 is the memory of the total energy of A so to speak. If C is positive it means the total energy is positive. In other words, the kinetic energy outweighs the potential energy. The galaxies are going to continue to recede forever. They have beaten the escape velocity.

Conversely, if $C < 0$, then the total energy of each galaxy is negative. Each galaxy has more negative potential energy than kinetic energy. That is the situation where we expect everything to go out away and then come back and crash.

So now equation (20) is our new equation for the evolution of the universe. It is the real Friedmann equation, valid even for non zero energy, that is for a Newtonian universe not at escape velocity. It was derived by Alexander Friedmann from general relativity not from Newton. Nevertheless we just derived it from Newton.

Let's examine equation (20), and see if we can figure what it says. It is not too hard to solve for any given case of the value of C. But it is a bit messy and we don't need to for our purpose which is to understand qualitatively the possible evolutions of the universe.

Let's assume that the universe began by growing – that is with $\dot{a} > 0$ – *and* that C is positive. Since everything else on the right side of equation (20) is strictly positive, it will remain so. Therefore $\dot{a}/a$ cannot change sign, because it would have to go through zero. In other words, the universe will continue to grow forever. It may slow down but it cannot stop and reverse course.

In the Newtonian model which we have been studying so far [4], $a(t)$ cannot grow asymptotically, as time goes on, to a finite value. Indeed $\dot{a}/a$ stays bigger than in the zero energy case. And in that case we showed that a grows to infinity. That must be the case here too.

So if C is positive, the universe will grow to arbitrarily big size. Consequently the scale factor a will also become arbitrarily big. This will enable us to figure out the motion of the universe as time goes on.

We shall follow a standard way of thinking that is used in all kinds of situations in physics : go to limits. Here we will use the fact, on the right-hand side of equation (20), that

4. In this Newtonian model, the universe is always infinite. When we say that it grows, we mean that the scale factor grows, that is that the galaxies, while staying homogeneously distributed on average in an infinite 3D Euclidean space, become more distance from each other, as shown in figures 3a and 3b. We will meet in lesson 3 other models of the universe with other spatial geometries.

the two terms C/a^2 and $8\pi G\nu/3a^3$ behave differently when a is small or a is big.

Let's start with small a first. When a is small which is bigger, $1/a^3$ or $1/a^2$? Answer : $1/a^3$. So for sufficiently small a, on the right-hand side of equation (20), the second term is negligible compared to the first [5]. We have already studied the case without the term C/a^2 and we know the answer : a grows like $t^{2/3}$. In other words, in the very early phases of the expansion of the universe, when a is just starting out growing, the scale factor follows the same curve as in the zero energy, exact escape velocity, case.

What about the case when a is very big ? Now it is the term in $1/a^3$ which is negligible compared to the term in $1/a^2$. We may have to wait a while, and how long we have to wait depends on constant C, but eventually the second term will become much larger than the first.

So let's study the equation that we obtain when we neglect the first term on the right of equation (20) [6]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{C}{a^2} \tag{21}$$

This reduces very simply to $\dot{a}$ equals a constant. That is, a has a constant slope. In other words, in the situation where the energy is positive, when the galaxy gets far enough away

5. Of course this model assumes that the Big Bang began with a scale factor close to zero, which is not a consequence of the simple reasoning followed so far in chapters 1 and 2.

6. As we most usually do in physics, we assume that equations close to each other produce solutions close to each other. With a bit of mathematics it can be made rigorous in the cases we are considering.

the effect of gravity becomes negligible and the galaxy just moves off with uniform velocity.

The graph of $a(t)$ is a combination of a curve proportional to $t^{2/3}$ at first and a straight line later on, with a more complicated shape in between. It looks something like this

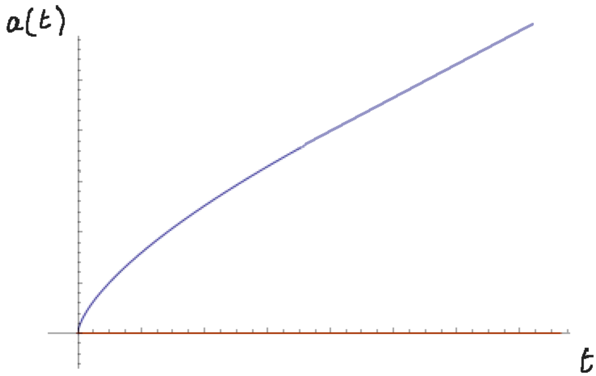


Figure 6 : Evolution of the scale factor $a(t)$ in the case of positive total energy.

Nothing deep is going on here for a big. When there is no gravity Galileo showed and Newton explained that things move with constant velocity. In the case of a projectile thrown away from the Earth with a velocity bigger than escape velocity, it will first of all slow down under the effect of gravity. But after a while, gravity will become negligible, the potential energy will be close to zero, the kinetic energy will remain constant and positive, and the projectile – say, an apple vigorously thrown away by Newton – will continue coasting along in the universe at constant speed.

Exercise 1 : Assuming as usual that $\dot{a}(t)$ starts out positive, show that, in the case of a negative constant C in equation (20), the graph of the scale factor $a(t)$ will have a shape like in figure 7.

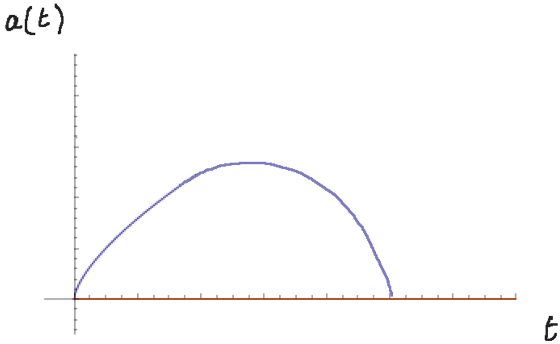


Figure 7 : Evolution of the scale factor $a(t)$ in the case of negative total energy.

That is the *matter dominated universe*, and its evolution when it is respectively above escape velocity (fig. 6) and below escape velocity (fig.7).

Why is it called the matter dominated universe ? Because the right-hand side of the Friedmann equation contains a variable ρ – or ν divided by a^3 – which is just the density of ordinary non-relativistic slowly moving matter. It was the classic cosmology until other things were discovered in late XXth century.

So there are three possibilities : positive energy, negative energy or zero energy. As a consequence there are three different behaviors.

Who decides whether the energy is positive or negative ? Who knows ? But what we are going to find is that this positiveness or negativeness is connected with the geometry of the universe. In the next lesson, we will talk about the connection of C in equation (20) with the spatial geometry of the universe.

It is the main thing at this stage that the general relativity brought to bear. The equations are exactly the same as those we reached, but the significance of the constant C takes on a new dimension and has to do with geometry.

The last topic of the chapter is what happens if, instead of being made out of material points slowly moving, the universe is made out of photons, that is of radiation.

Radiation-dominated universe

To understand how does evolve a universe made out of photons, of course we really have to think about relativity. But there is only one important idea to use. It is $E = mc^2$.

If we had worked the Einstein's way, as in volume 4 of the collection *The Theoretical Minimum* on general relativity, we would have reached the same Friedmann equation as above, see equation (20).

In general relativity it is usual to write it as follows

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{C}{a^2} = \frac{8\pi G}{3}\rho \tag{22}$$

because in this presentation the two sides have an interesting interpretation [7].

Einstein's equation, which we established in chapter 9 of volume 4 on general relativiy, was a tensor equation with on the left-hand side the Ricci tensor minus one half the metric tensor times the scalar curvature, and on the right-hand side $8\pi G/3$ times the energy-momentum tensor [8]. In other words, as the reader remembers, Einstein's equation relates geometry of the universe, on the left, to sources of energy and momentum in the universe, on the right.

Equation (22) comes from Einstein's equation when looking at a simple model and only at one component of the tensors. Indeed $(\dot{a}/a)^2$ clearly has to do with the geometry of space. Moreover we will see in the next lesson that the constant C, in C/a^2, has to do with the curvature of space. And what is the energy on the right-hand side ? It is essentially the masses of the points as they appear in $E = mc^2$.

That is the connection between Friedmann's equation and general relativity. We derived Friedmann's equation in a classical Newtonian non-relativistic framework, in which we considered masses moving at slow speed with respect to

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7. Remember that C, in equation (22), is not the speed of light. It is just a constant.

8. In the calculations in volume 4, the factor 3 in the denominator was included in the energy-momentum tensor and did not appear explicitely.

each other when they were close. And we applied to them Newton's laws.

The only thing that we really have to remember, now that we want to consider a universe made of photons, is that when we go from Newton to relativity what was mass density becomes energy density.

All forms of energy density – not only mass density – is what goes on the right-hand side of equation (22) now. To be accurate it is energy density divided by the square of the speed of light. But as usual we will work with units such that the speed of light c is equal to 1.

With that idea in mind, ρ on the right-hand side of Friedmann's equation, instead of being the mass density becomes the total energy density. Some of it comes from the $E = mc^2$ kind of energy of the particles at rest. Some of it might be kinetic energy of particles. But if the universe is filled with photons then the energy we are really concerned with is radiation energy.

So let's talk about radiation energy and how it is different from the $E = mc^2$ energy.

We shall build a reasoning tool enabling us to analyse how the energy of photons evolves. To that effect, let's think of a box corresponding to one unit cube in the grid. It is a box whose actual side dimension increases like $a(t)$. For instance think of the box formed by the planes, $X = 0,\ X = 1,$ $Y = 0,\ Y = 1$, and $Z = 0,\ Z = 1$ in grid coordinates. As usual we represent it in two dimensions, figure 8, but think of three. The volume of the box is $\mathcal{V} = a(t)^3$.

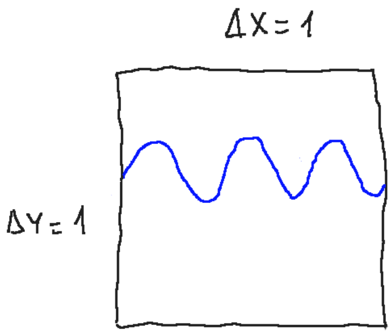

Figure 8 : Box of one unit grid-volume, containing photons. Represented is one photon, moving horizontally with a certain wavelength. But think of many photons moving in all directions.

The box may contain massive particles which don't move with respect to the expanding grid (that is how we constructed the grid). And we saw that their total energy doesn't change. Since, when considering their Euclidean coordinates, they move at non-relativistic speed, like a, their energy consits mostly of their mass, according to the formula $E = mc^2$. And since we set $c = 1$, the formula for the energy of a massive particle is $E = m$.

The box also contains photons. We represented one in figure 8. But let's suppose it contains many photons moving in all directions. The analysis of the energy of photons is quite different from that of massive particles.

The first thing to know about a photon, or electromagnetic radiation, is that its energy is related to its wavelength. For the energy of a photon the formula is

$$E = \frac{hc}{\lambda} \tag{23}$$

where h is Planck's constant, λ is the wavelength of the photon, and c is the speed of light, which as said we will set equal to 1. The important thing to remember about the energy of a photon is that it is inversely proportional to its wavelength.

Something which I am not going to prove, but I'm asserting, is this : consider a box – to start with, a box with walls against which anything moving inside it bounces, staying in the box. Suppose the box contains a photon of a given wavelength at time t. And we expand the box slowly. Then the wavelength of the photon will increase. It will just stretch in proportion to the actual side dimension of the box. Inversely, its energy will decrease.

We will apply the reasoning to grid boxes in the universe. The universe is expanding pretty slowly, since it takes 10 billion years for it to double in size, which is pretty slow. The fictitious boxes, moving with the grid, in the universe have no walls, but since the density of photons is homogeneous, as was matter in the matter-dominated model, on average for any photon leaving there is a photon entering, and it is the same as if there were walls.

We won't prove rigorously the photon's wavelength increases, but we can explain it heuristically. Anybody who plays the guitar knows the phenomenon very well. The box with walls is replaced by the string of the guitar. The string is pinned at one end by the bridge of the guitar. At the other end it is pinned by your finger on a fret. When you pluck the string it starts to vibrate, see figure 9.

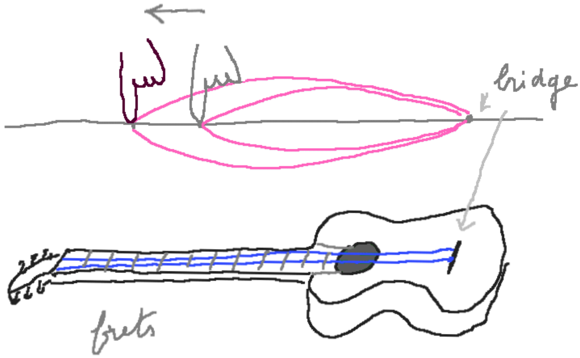

Figure 9 : Guitar string vibrating at different frequencies when we move our finger on different frets.

What happens as you slide your finger ? If you slide it to make a longer string the wavelength goes up and the tone, that is the frequency of vibration, goes down. Doing so, you are effectively changing the size of the box. Changing the size of the box, the wavelength changes.

The same thing happens to radiation in a box. The radiation wavelengths just stretch in proportion to the size of the box. Because the wavelengths change for the photons, that means the energy of each photon changes as we change the size of the box.

The world is filled with photons. As explained above, we can consider that the number of photons in each fictitious grid-box stays fixed. But their energy changes as we change the size of the box. More precisely, the energy of each photon will be proportional to 1 divided by the size of the box

$$E = \frac{h}{a} \tag{24}$$

In summary, we look at one fictitious grid-box, let the scale factor of the universe grow larger, then the energy of each photon decreases. It is the same phenomenon as the frequency of the guitar string vibration going down as we make the vibrating part of the string longer.

We would have to do a little more quantum mechanics or a little more classical electromagnetism in the presence of an expanding universe to justify properly the conclusion that the wavelength of the photons stretch proportionally to the scale factor as the universe expands. Let's leave it at that in this lesson, and focus on the consequences of it.

The main consequence of it is that the energy per photon decreases like 1 over a, in contrast to the case with ordinary particles where their mass stays the same as time goes on. So, unlike in the ordinary massive particle case, the total energy in the box – which is also its mass – does not stay constant. The total energy in the box decreases like 1 over a.

Now, compared with the previous case, there's one more factor of a in the denominator on the right-hand side of equation (22). The energy density was

$$\rho = \frac{\nu}{a^3}$$

that is, it evolved like the inverse of a cube. Now it evolves like the inverse of a to the fourth power

$$\rho = \frac{\nu}{a^4}$$

Remember that ν is the total energy per unit grid-volume.

It is the only new thing that happens :

a) in the *matter-dominated* universe ρ goes like $1/a^3$

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{C}{a^2} = \frac{8\pi G}{3}\frac{\nu}{a^3} \tag{25a}$$

b) in the *radiation-dominated* universe ρ goes like $1/a^4$

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{C}{a^2} = \frac{8\pi G}{3}\frac{\nu}{a^4} \tag{25b}$$

In both cases the left-hand side, in other words the geometry side, of Friedmann's equation stays the same.

Let's study equation (25b) in the case where $C = 0$, as we first did in the matter-dominated case, just to see the difference. Again, by appropriate choice of the size of the grid we can rearrange it so that on the energy side all the constants reduce to 1. We simply get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^4} \tag{26}$$

Exercise 2 : Solve equation (26).
Hint : use the same trick as before, which was to interchange a and t as the dependent and independent variables.

We get, up to a proportionality constant which is not important and we disregard, the following relation

$$a = t^{1/2} \tag{27}$$

In the matter-dominated zero total energy case we had reached $a = t^{2/3}$. The two curves are represented below

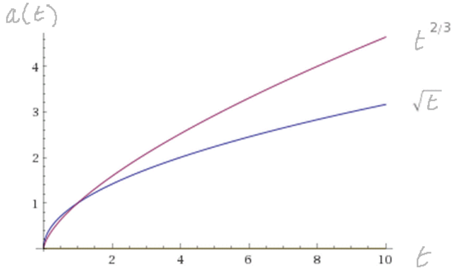

Figure 10 : Evolution of the scale factor in the case $C = 0$, in a matter-dominated universe and in a radiation-dominated universe.

Before $t = 1$ the square root is above the two-thirds power, then it is the other way around. But our time units have no particular physical meaning here. Qualitatively the two curves look similar. Of course if you are an astronomer and really want to know what is going on in the universe, the difference between t to the one-half and t to the two-thirds can be very important.

What about the mixed case ? Suppose the universe has both ordinary particles and radiation, like the real universe really does. Our universe indeed is neither pure radiation nor pure non-relativistic particles.

In that case the energy density has two components, one for radiation that goes like 1 over a^4, and one for ordinary matter that goes like 1 over a^3. Staying in the case where the constant C, on the geometry side of the equation, is equal to zero, Friedmann's equation now has the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{C_m}{a^3} + \frac{C_r}{a^4} \tag{28}$$

where C_m and C_r are two constants, both positive. C_m is just some measure of the number of ordinary massive particles in a unit grid-box. And C_r is a corresponding measure of the number of photons.

Equation (28) is the equation of motion for a universe containing, like the real universe does, ordinary non-relativistic matter plus radiation.

Which term, on the right-hand side of equation (28), is the more important ? Answer : it depends on a of course. When $a(t)$ is big C_m/a^3 is the more important. For small $a(t)$ on the other hand, C_r/a^4 is the more important. And the smaller the a, the more more important it is.

So what that tells us without too much work is that when a is small, in the beginning of the expansion, the only thing that is important is the radiation. At first, the radiation term is dominant compared to the material term, protons, neutrons, galaxies... Well, there were no galaxies at the beginning of the universe. But anyway at the beginning the radiation was the most important thing.

Therefore the universe started expanding like $t^{1/2}$. But eventually the term C_m/a^3 took over, then the scale factor continued to grow like $t^{2/3}$.

Question : how do we know that no energy of matter gets converted to energy of radiation ? That is something we are going to look at in more detail. The short answer is that when things get cold enough, when the universe has expanded enough, there is no longer much exchange between radiation and ordinary matter. They are pretty well conserved each one on their own. The cooling point at which matter and radiation cease to interact is a rather high temperature of more than ten thousand degrees.

Once things have cooled down to a certain temperature, the photons are pretty much free streaming and don't care about the particles. The photons have then too long wavelengths to even scatter with the particles very much [9]. So in this model-universe with mixed matter and radiation, we assume that the two constants C_m and C_r are essentially time independent.

In this model there are only two components : matter and radiation.

The radiation component is made of several different things. There are photons of course. There are presumably gravitons. And there are neutrinos. Neutrinos do have mass, but it is so tiny that even today the neutrinos are moving with very close to the speed of light, and simply mimic the same behavior as the photons.

9. The basic example of scattering of a photon by a particle is the Compton effect, discovered by American physicist Arthur Compton (1892 – 1962) in the early nineteen twenties. It is an inelastic shock so to speak between a photon and a free electron, where the outgoing photon has a different wavelength from the incident one.

To summarize, in equation (28) there is the radiation component C_r/a^4 which consists of all the particles so light that they are moving with close to the speed of light. And there is the mass component C_m/a^3 which consists of all particles heavy enough that they are basically at rest relative to us nearby. That is cosmology as it was known up to the nineteen eighties.

Cosmologists now add a third component, it is the dark energy discovered in late XXth century. We will talk about it in chapter 3.

Thus there will be three categories :

1. ordinary matter,
2. radiation,
3. dark energy.

The first two are the commonplace things we have amply described.

Dark matter, which the reader has probably heard of too, should not be confused with dark energy. It belongs to the matter category. It is a part of the ordinary particle matter. It is invisible simply because it doesn't have any charge. It doesn't radiate much. So we don't see it optically. It is part of the constant C_m.

The model-universe filled with photons whose wavelengths increase with the scale factor raises one question : we saw that their energy decreases ; where does it go ? Answer : if we consider first of all a real expanding box with walls, the photons do work on the walls. They exert pressure, like light on those funny little lightmills sealed with rarefied atmosphere in a glass, which we can buy in gadget stores, and which start to spin when lit, see figure 11.

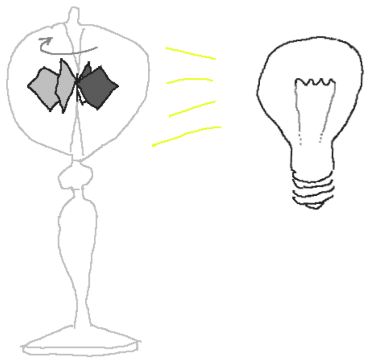

Figure 11 : Lightmill.

So the photon pressure on the walls of the box does work. This energy can go to a number of places, even when considering a box which doesn't lose energy to the exterior. It could go into stretching the distance between the molecules forming the walls – the links equivalent to springs holding the wall molecules together according to Hooke's law, which hold potential energy. It could cause the walls of the box to gain kinetic energy, or whatever else, just heating the box for instance. One way or another it goes into increasing the other energies of the box itself.

For the whole universe, whose motion is described by equation (28), the energy lost by the photons goes into the kinetic energy of massive particles related to the expansion of the universe. It changes the term $(\dot{a}/a)^2$ which comes – the reader will recall – from considering the kinetic energy of galaxy A in figure 4, and then doing some algebraic manipulations on the equation expressing that the total energy was zero. All our equations come from applying the energy conservation principle, so there is no way that that we can be violating energy conservation.

Of course, our Newtonian model of infinite Euclidean universe has limitations. So far, even though we made a mild reference to general relativity when we said that Friedmann's equation (22) can be derived from Einstein's equation, we have been investigating classical Euclidean Newtonian models that in truth work well only when looking at comparatively small regions of the universe – just like around us we can do physics as if the Earth was flat, but not if we consider a phenomenon taking place in a region 5000 km wide for instance, like sending a ballistic projectile. The various limitations of our models clear up when using full fledged general relativity. See the questions / answers section at the end of the chapter for a further discussion of this point.

We can cover a good deal of cosmology, however, without all the apparatus of general relativity, tensor algebra, funny metric, Minkowski-Einstein geometry and the like, that are the subject of volume 4 in the collection *The Theoretical Minimum*. So we will continue for a while to study Friedmann's equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{C}{a^2} = \frac{8\pi G}{3}\rho$$

and its extensions using only light means.

The constant C has to do with the fixed total energy of the universe. It also has to do with the curvature of space – not spacetime, just space.

We are going to study three cases corresponding to three interesting geometries. Let's temporarily think of the spatial universe, setting time aside, as a two-dimensional variety to support our geometric intuition.

1. There is a flat geometry where space itself is flat. Einstein's general relativity does not forbid flat space. The space – a plane when we are in 2D – goes on and on endlessly, homogeneously, and triangles have the usual properties. The sum of their angles is 180 degrees. That is the case where $C = 0$.

2. There is the case where the universe is curved like the surface of a sphere. If you walk straight ahead, you eventually come out from the other side. It is finite, and compact we say. That is the case where C is a positive number. It corresponds to the radius of curvature of the expanding universe. It is easy to visualize how a flat plane can be replaced by a positively and uniformly curved two-dimensional space, namely the surface of a sphere. An analogous operation, hard to visualize but easy to characterize algebraically, enables one to go from a flat 3D Euclidean space to a positively and uniformly curved three-dimensional space.

3. Finally there is the case where C is a negative number. It corresponds to a negatively curved space. We shall have to explain in detail what it is like. It is

less easy to visualize than the surface of a sphere.

These will be the topics of the next chapter.

Questions / answers session

Q. : How can we study a model-universe with a non zero constant C, therefore a non zero curvature, if the general framework we use is a Euclidean space the curvature of which is zero ?

A. : Yes, among the paradoxes we mentioned about our Euclidean models is that a 2D or 3D Euclidean space is flat, yet in it we are bold enough to study model-universes where C is not zero. In truth we only sort of mimic a curved space if you will. It is a bit of a fake. But as said we can make quite a lot of interesting deductions investigating this model without using full fledged general relativity.

Q. : How soon after the Big Bang did the universe cool down to a point where the matter and radiation components were not really linked anymore ?

A. : About a hundred thousand years.

Q. : What astronomical evidence do we have to corroborate growth like $t^{1/2}$ in the early stage of the universe ?

A. : We are talking about the expansion history of the universe. Remember that looking out at the universe we effectively see a history. Looking to different distances we are seeing the universe at different times. And by looking at different times – assuming that it is homogeneous – we can reconstruct the history of expansion. We will talk about that. Basically looking at a combination of density of galaxies in the sky, careful measurements of distances of galaxies, the Cosmic Microwave Background, and so forth, allows us to reconstruct the history of expansion. We also reconstruct the temperature history of the universe.

There is plenty of evidence that the picture of the initial growth like $t^{1/2}$ is right. It was before galaxies formed.

The later $t^{2/3}$ part, in the past, is easy to see. The early $t^{1/2}$ has largely to do with the Cosmic Microwave Background and things like that.

We will get the equations down and then will talk about how much of the picture is right.

The answer is : it is right for the early stages of the universe but it is wrong for the present times. We know that at present the universe is growing faster than $t^{2/3}$, see figure 12.

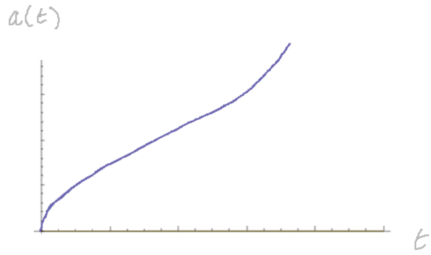

Figure 12 : At present the universe is growing faster than $t^{2/3}$.

Q. : In the Newtonian model we built there are objects that are moving relative to each other at faster than the speed of light c. Isn't it a principle, though, that nothing can go faster than the speed of light ?

A. : Well the principle really says that you cannot see anything going *past you* faster than the speed of light. And you cannot send messages that will reach their destination faster than c. But it doesn't say that two objects in an expanding universe cannot be receding from each other at faster than c. That is quite allowed.

Indeed in our model of universe growing like $a(t)$, two massive objects separated by a distance $D = aX$, where X is a fixed grid-distance, will have a relative speed $V = \dot{a}X$. This gives the relation

$$V = HD \tag{29}$$

where H is the Hubble constant $\dot{a}/a$. (Remember that H depends on time but does not depend on space. It is the same for any pair of objects.) So at any given time t, if we choose D large enough, that is two galaxies distant enough from each other, then we surely can make V higher than the speed of light c.

It is connected with the idea of the horizon. But you are right : these equations, $D = aX$ and $V = \dot{a}X$, say that things can recede from each other at faster than the speed of light.

Q. : Does it mean that we can see an object accelerating away from us as long as it goes slower than the speed of light, but when it eventually goes faster than the speed of light relative to us, it will disappear ?

A. : Yes, when it reaches the speed of light relative to us it will go through the horizon and we won't be able to see it anymore.

Actually that has to do with dark energy. Without dark energy nothing would really move out of our ability to see them. I will show you how the geometry works. And I will show you how all these things can be understood. But let's go one step at a time.

Q. : What is the relationship between the evolution of $a(t)$ and what we can see ? Where is the horizon for a given object ?

A. : We just recalled that, in our model-universe, equation (29) tells us that at any given time t there are objects receding from us at faster than the speed of light.

But if $a(t)$ grew really like $t^{2/3}$ the expansion would slow down, because that would mean

$$\dot{a}(t) = \frac{2}{3} \frac{1}{t^{1/3}} \tag{30}$$

So if we considered a given object, at a grid-distance X from us, it might during a certain period of time go faster than c, but it would eventually go slower than the speed of light. Therefore it would become visible again.

So if the pattern of growth was $t^{1/2}$ followed by $t^{2/3}$, as derived from equation (28), eventually we could see everything no matter how far away.

But it is not the pattern of growth of the universe we observe. What we observe is shown in figure 12. And if this pattern with an acceleration is correct, then there is an ultimate limitation, so that we cannot see past a certain point.

Let's hold back on that. We are going to spend a lot of time on those particular things. They are the really interesting things and we will explore them.

Q. : Can you explain a little more how, in the radiation-dominated universe, the wavelengths of the photons decrease ?

A. : Ok, the question can be reformulated as this : why is the wavelength of the photon sort of rigidly attached to the grid, if you like ?

Anybody who knows a little bit about waves and proper wave propagation knows that there is something called an adiabatic invariant. It is the number of nodes, that is the number of times the wave passes through zero. As long as you change things slowly the number of nodes stays fixed. That is the adiabatic invariant.

Consider a closed loop, for instance a circle, on which a wave is propagating, see figure 13.

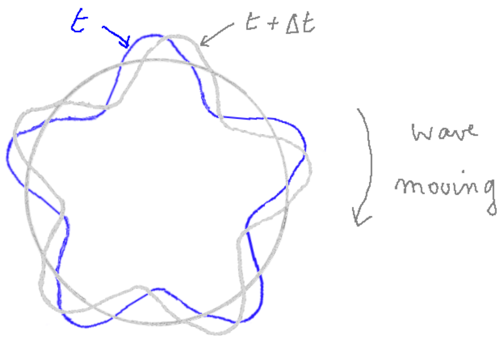

Figure 13 : Wave propagating on a circle. We could also consider a standing wave, the reasoning would not change. We could even consider a wave packet of any shape somewhere on the circle instead of a nice uniform wave.

This is something that in principle we could build. The wave represented has 10 nodes. Now suppose there is a device enabling us to increase slowly the radius of the circle, and therefore the length of the circonference. Will the wave at some point change its number of nodes ? No. The number of nodes will stay the same, and – as long as we proceed gradually, continuously, slowly –, instead, the wavelength will increase. The nodes number must be an integer, and it cannot jump.

It is the same phenomenon if we replace the ring in figure 13 by the universe itself. Its expansion is slow – 10 billions years to double in size –, so it is adiabatic. Therefore the number of nodes per grid-distance of a photon doesn't change. Instead, its wavelength increases according to $a(t)$.

The reasoning on the ring remains valid if, instead of a nice uniformly spread wave, we consider a wave packet, non uniform in space, somewhere on the ring. The packet may be moving or standing, it doesn't make any difference either. Using Fourier analysis, we can decompose the wave packet into a collection of nice uniform waves of various number of nodes. And, as the ring inflates, those Fourier elements will keep their number of nodes. Their individual wavelengths will increase instead, and so will the wave packet.

Q. : As we look out into the universe and, at any given time t, consider galaxies which move faster as they are farther away from us, relativity says that their energy would increase. If we were to weigh them, it seems that we would get boundless masses. How is that possible ?

A. : Well you can't weigh something that is far enough away. But it is a good question.

To begin with, let's straighten terminology : *mass doesn't change*. What we call mass nowadays is what used to be called mass at rest. The mass of the electron is the fixed number 9.11 x 10^{-31} kilograms. When we want to talk about the quantity which increases with speed, we must talk about energy. And indeed, in our simple Euclidean model-universe, the kinetic energy of those distant galaxies, in our referential, grows without bound.

But the general point here is that we cannot analyze those kinds of questions by just thinking in terms of Euclidean Newtonian or even special relativity geometry.

We must specify the metric, the geometry of the universe, and then analyze them in that geometry. We will do that. We can go only so far without introducing real relativity. And by real we mean general relativity.

We didn't have to introduce relativity because, in our analyses, we implicitely stuck to a small enough region. All the equations were derived by looking at galaxies in a Euclidean Newtonian universe. That is correct if we restrict ourselves to a small region of the universe around us. The entire spatial universe may be curved but it is locally flat. And when the galaxies are nearby they move, with respect to us, with a very tiny fraction of c. So the particles we looked at never moved anywhere near the speed of light.

If we want to study the whole universe, and be able to talk about points such that the Hubble constant times the distance from us, cf. equation (29), is comparable or above the speed of light, we cannot do it without general relativity.

Therefore these questions are jumping ahead of the game.

Q. : In the radiation-dominated universe in expansion, we saw that the wavelength λ of a photon increases like $a(t)$, and therefore, according to the formula $E = hc/\lambda$, that its energy decreases. But if we look at an individual photon, where does this energy go ?

A. : The easiest is to start thinking about a single box with real walls and expanding. Photons push on the walls and, because the walls move, the photons lose energy. It is transfered to other forms of energy in the box. Then think of the entire universe as made of grid-boxes of side one in grid-coordinates, and with walls. Then remove the walls. Because the universe is homogeneous, the number of photons per box doesn't change. The energy of a photon does indeed satisfy $E = hc/\lambda$. And λ does vary like $a(t)$, as we amply explained. So the energy of radiation in the universe must go to other forms of energy that we will study in this course.

In fact the loss of energy of the photons is not related to the light character of these particles, their wavelength and so forth. The reader has certainly experienced it while pumping air in the tyres of his or her bike. Consider a chamber with gas in it and a piston as in figure 14.

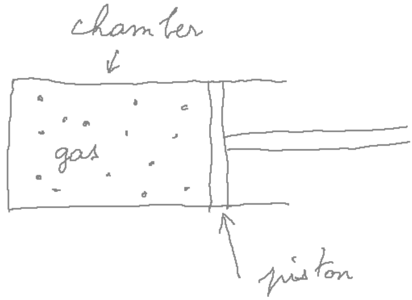

Figure 14 : Compressing or expanding gas in a chamber. If the experiment is done adiabatically, the gas heats up when the piston is pushed to the left, and it cools off when the piston is moved to the right.

When we push the piston to the left the gas heats up. And when we move it to the right it cools off. The experiment must be done adiabatically, which means with no exchange of heat through the walls of the chamber or the piston, and slowly. See volume 6 of the collection *The Theoretical Minimum* on statistical thermodynamics for a complete analysis of the phenomenon.

To move the piston slowly is important. We must leave time to the particles to hit many times the piston. Indeed if we moved the piston to the right extremely suddenly, and expanded say be two the volume of the chamber, then very temporarily the gas molecules would not have had time to move. They would all still be in the left side of the chamber. Neither their energy nor their momentum would have changed. That is a non-adiabatic change.

On the other hand, slowly means that we leave plenty of time to molecules to bounce off the walls many times. Then, during the expansion, the average kinetic energy of the molecules will go down. And the average kinetic energy of the molecules is the temperature. So the gas will cool off.

If you do things very suddenly it is likely to heat up whether you compress or expand it. At least it won't cool off.

So the decrease of photon energy as the universe expands is a phenomenon we are familiar with with ordinary gases. In the radiation-dominated universe you may call it photon cooling.