

## Lesson 5 : Vacuum energy

*Notes from Prof. Susskind video lectures publicly available on YouTube*

## Introduction

In this lesson, we shall go deeper into the equation of state and what it says about the evolution of the universe. We shall examine several versions of it, see where we get our knowledge about them – not the cosmological knowledge, but the knowledge derived from basic physics.

Before we go into that, however, let's glance at where we are, how we got there, and where we are going.

The basic question that we want to answer in cosmology is : what is the history of the universe? And to the extent that the universe can be thought of as isotropic and homogeneous, it really boils down to what is the time history of the scale factor.

If we know the time history of the scale factor, we know an awful lot about the history of the universe. We can test it. We can observe it in various ways. It is not the only question, but it is one overriding question that, if you want to do cosmology, you better have under your control. So, what is  $a(t)$  as a function of time? How does it evolve?

We studied two models. There was the matter-dominated model. In it  $a(t)$  expanded like  $t$  to the two-thirds. And there was the radiation-dominated model, where  $a(t)$  expanded like  $t$  to the one half, that is  $\sqrt{t}$ .

$$\text{a) matter-dominated : } a(t) \sim t^{2/3} \quad (1)$$

$$\text{b) radiation-dominated : } a(t) \sim t^{1/2} \quad (2)$$

Both are *models*. Neither of them is exactly correct. Today at late times, the matter-dominated model is almost exactly

correct. At very early times, we believe that the radiation model was more correct. And there was a transition between them.

When we get to observational cosmology, we are going to talk a great deal about how we know anything about equation (1) or equation (2), and what the various meanings of them are. But this will be in future lessons.

We also talked about the equation of state. The radiation-dominated and the matter-dominated universe are two examples of universes evolving under different conditions which can be characterized by an equation of state.

The equation of state is a thermodynamic equation leading us to how the energy density  $\rho$ , which appears on the right-hand side of the Friedmann equation, changes with changes in the scale factor. Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad (3)$$

For example, in the matter-dominated case it lead us to  $\rho$  equals some constant – let’s call it rho naught – divided by  $a$  cube.

$$\rho = \frac{\rho_0}{a^3} \quad (4)$$

$a^3$  is the volume of a piece of space as it expands. The energy density is the amount of energy in it divided by the volume. So that is just something over  $a^3$ .

In the radiation-dominated case, which we are going to talk about extensively in this chapter, the analog of equation (3) is

$$\rho = \frac{\rho_0}{a^4} \quad (5)$$

We are going to examine in detail, in this radiation case, where the equation of state comes from, and how we use it to establish equation (5).

In cosmology, the *equation of state* is the name of the equation expressing the relationship between pressure and energy density. It is a particular example of the equation of state of any statistical mechanical system, which usually involves other variables like temperature, see volume 6 of the collection *The Theoretical Minimum*, on statistical mechanics.

But in cosmology we make the simplifying assumption that the energy density and the pressure are simply related, and that the temperature doesn't appear in the equation, nor any other thermodynamic variable. This is for the sake of simplicity and also because it covers a lot of interesting cases. So it is simply

$$P = w\rho \quad (6)$$

$w$  is a number. It characterizes the fluid making up the material in the space, whatever it happens to be. And the equation of state (6) expresses that the pressure  $P$  is equal to  $w$  times the energy density  $\rho$ .

Because it is important to understand very well the physical and mathematical reasonings, already exposed in the

previous lesson, that took us from the equation of state to equations (1) or (2), relating the energy density and the scale factor, let's go over them quickly once more.

We start with the equation of state for the matter-dominated case. A matter-dominated universe is a universe made of particles, protons, neutrons, electrons, atoms, galaxies... It is non relativistic matter, that is stuff which, in its own local frame, is moving slowly<sup>1</sup>.

What is the energy density? The energy density is the energy in a volume divided by the volume. And since the universe is assumed to be homogeneous, the density is the same everywhere. The energy in a volume is that of the particles inside. For any particle, it is just the  $E = mc^2$  energy of the particle at rest, plus its ordinary kinetic energy, plus negligible higher order terms<sup>2</sup>

$$E_{particle} = mc^2 + \frac{1}{2}mv^2 + o(v^2) \quad (7)$$

If we are thinking about particles which are all moving slowly, the huge factor  $c$  squared in the first term on the right makes the other terms negligible. A tiny dust grain, because of its mass, has enough energy to cause a big explosion if it were annihilated, even though its kinetic energy is insignificant.

Where does the pressure come from? If we were thinking about an ordinary gas in a volume of space bounded by

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1. By that we mean that in the frame of any particle, the other particles locally are moving slowly.

2. See chapter 3 of volume 3 of the collection *The Theoretical Minimum*, on special relativity and electrodynamics.

walls, the pressure on any wall would come just from particles hitting the wall. It is proportional or related to the velocity of the particles which hit the wall. The formula for the pressure contains the mass. This factor mass is important. If a bowling ball hits the wall it creates a bigger force on the wall than a Ping Pong ball does.

But whatever is hitting the wall is hitting it slowly because all the particles are moving slowly. The formula for the pressure does not contain the speed of light in it. It contains the velocities of particles instead. Because it doesn't contain the speed of light, typically the pressure for ordinary non-relativistic particles is negligible compared to their energy density.

That is the approximation leading us to say that the pressure is zero compared to the energy density. Therefore  $w$  is equal to zero in equation (6). So for non-relativistic matter density  $w = 0$ .

On the other hand, for radiation  $w = 1/3$ . We will prove later in this chapter.

But, first, let's just remind ourselves how we use the equation of state. One of the things that we need to know in order to work out the equations of cosmology is how the energy density depends on  $a$ . How we go from  $P = w\rho$  to equation (4) or equation (5) expressing  $\rho$  in terms of  $a$ .

We began with a box of gas, figure 1.

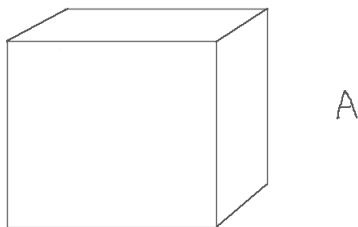


Figure 1 : Box of gas.

The equation of state is not an abstract relationship between abstract variables<sup>3</sup>. It can be analyzed by laboratory methods. Of course if the gas that makes up the universe is made up of galaxies it is not so easy to put a bunch of galaxies in a box. But galaxies are just particles from our point of view. You can put particles in a box. And in the box you can investigate the relationship between the energy and the pressure. And that is what we are interested in.

So we take a box containing gas at a certain pressure. This pressure pushes on all the walls<sup>4</sup>. Let's consider for instance the wall on the right in figure 1. The pressure exerts a force on that wall.

$$P = \frac{F}{A} \tag{8}$$

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3. Like is, for instance, the entropy  $S$  in  $dS = dQ/T$

4. Notice that the pressure exists anywhere inside the box too. But on any two-dimensional piece of wall we could put inside the box for an experiment, two forces opposite to each other would be exerted. Similar considerations explain why you can fill a glass of water up to the brim, place a flat piece of Bristol cardboard on it, turn the system upside down, and the cardboard and water don't fall.

where  $P$  is the pressure,  $F$  is the force, and  $A$  is the area of the wall. Now let's suppose we expand the box a little bit, as shown in figure 2.

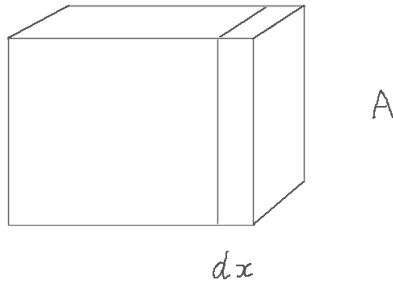


Figure 2 : Increase in volume  $dV = A dx$ .

What happens to the energy inside the box? If the gas inside the box is exerting pressure on the wall and we move the wall, then the gas does some work on the wall. Work is equal to force times distance. So there is a little bit of work that is done. It is given by

$$F dx = P A dx \quad (9)$$

That is, the work is the force  $F$  times the little displacement  $dx$ . And that is also equal to the pressure  $P$  times the area  $A$  of the wall, times  $dx$ . Now  $A dx$  is the change in volume. So equation (9) can be rewritten

$$F dx = P dV \quad (10)$$

That is a famous equation in thermodynamics. I just wanted to write it down because I like it.



So what happens to the energy in the box? The gas has done some work. If the gas does work then the energy in the box must decrease. Generally speaking, if something does work then its energy decreases.

It means that the change in energy in the box must be

$$dE = -P dV \quad (11)$$

That is our equation. Let's go through it quickly to remind ourselves how it tells us anything about how energy density scales with the scale factor. We remember that the energy inside the box is given by

$$E = \rho V \quad (12)$$

So we can also compute  $dE$  using ordinary calculus

$$dE = \rho dV + d\rho V \quad (13)$$

Both  $\rho$  and  $V$  change, in general, when we expand the box a little bit. The energy density changes, and certainly the volume changes. The net change in the energy is the sum of the two terms on the right-hand side of equation (13). And, from equation (11), that has to be equal to minus the pressure  $P$  times  $dV$ .

$$\rho dV + d\rho V = -P dV \quad (14)$$

Now let's plug-in the hypothetical equation of state. It is an equation we haven't really justified yet. But let's plug-in our guess for the equation of state.

$$P = w\rho \tag{15}$$

So we just substitute the number  $w$  times the energy density for  $P$  in equation (14).

$$\rho dV + d\rho V = -w\rho dV \tag{16}$$

Then we put all the terms with  $dV$  on one side, and all the terms with  $d\rho$  on the other.

$$V d\rho = -(1 + w)\rho dV \tag{17}$$

It is the preliminary to another famous equation. Then we divide by  $\rho$  and by  $V$  to get all things with  $\rho$  on one side, and all things with  $V$  on the other.

$$\frac{d\rho}{\rho} = -(1 + w) \frac{dV}{V} \tag{18}$$

It is getting more famous, but it is not quite famous yet. Remember that  $(1 + w)$  is just a number, whatever it is.

The next step in our reasoning is :  $d\rho/\rho$ , that is the differential of the logarithm of  $\rho$ , and similarly  $dV/V$  is the differential of the logarithm of  $V$ . So

$$\log \rho = -(1 + w) \log V + c \tag{19}$$

where  $c$  is a constant. Or

$$\rho = \frac{c}{V^{(1+w)}} \tag{20}$$

where  $c$  is another constant, noted with the same generic letter.

Now equation (20) is a famous equation. It says that the energy density  $\rho$  is inversely proportional to the volume of the box  $V$  raised to the power  $(1 + w)$ .

Finally we link this to the scale factor  $a$ . To push through the equations from (8) to (20), we increased the box along only one axis. But we could have expanded the box uniformly along all three axes – that is, we could have expanded it isotropically –, we would have gotten exactly the same equation (20) between  $\rho$  and  $V$ .

When the scale factor  $a$  increases, the dimension of the box *along each axis* increases proportionally to  $a$ . So the volume of the box increases in proportion to the cube of the scale factor :  $V \sim a^3$ . Thus we have

$$\rho = \frac{c}{a^{3(1+w)}} \quad (21)$$

where  $c$  is yet another constant, which we could call  $\rho_0$ .

If you are one of those crazy people who like to do cosmology in different numbers of dimensions, then in four dimensions the  $a$  cube in equation (21) would become the 4th power of  $a$ . In two dimensions it would become  $a$  squared, and so forth. But otherwise it would be the same. If you are a sensible three-dimension person, equation (21) is the formula. And this formula is famous.

Equation (21) is valid for any kind of material inside the box : gas, liquid, photons... We did not have to know spe-

cifically which kind of substance caused the pressure, the force, the variation of energy, etc. It is also true for the whole universe filled with atoms, galaxies, black holes, and also radiation.

Let's go again to the matter-dominated case where the pressure is almost zero because things are moving slowly. If the pressure is almost zero, and the equation of state  $P = w\rho$  is correct, it necessarily corresponds to  $w = 0$ .

In that case, equation (21) becomes

$$\rho = \frac{c}{a^3} \quad (22)$$

which is the same as equation (4) above.

In the case of a radiation-dominated universe, so far I just told you that  $w = 1/3$ . But we shall prove it in this lesson. Then in the denominator of equation (21), the exponent of  $a$  is  $3(1 + 1/3) = 4$ , therefore equation (21) becomes

$$\rho = \frac{c}{a^4} \quad (23)$$

It is the same as equation (5) above.

That was a long review to make sure we understand well what is the equation of state, and how we use it to find the relationship between the energy density and the scale factor, which in turn enable us to solve Friedmann equation.

Now let's come to the question of why  $w$  is equal to  $1/3$  for radiation.

## Equation of state in a radiation-dominated universe

In this section we shall prove that the value of  $w$ , in the equation of state  $P = w\rho$ , in the case of a radiation-dominated universe, is  $1/3$ .

Radiation is massless particles, that is, photons. We could think of it also as electromagnetic waves. We would get the same answer, incidentally, using the physics of waves. But let's think of it as photons.

The characteristic feature of photons that makes them different from non-relativistic matter is that the photons are moving fast. In fact they are moving with the speed of light. So let's work out in detail the equation of state for a box filled with photons. Again we represent a box, figure 3.

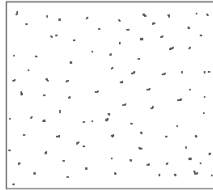


Figure 3 : Box filled with photons.

It is still a 3D box. But it is easier to draw it 2D, and the forthcoming reasoning will be easier to illustrate with a two-dimensional picture.

The box is filled uniformly with lots of photons. Of course figure 3 is an instantaneous snapshot, because we should

think of the photons as whizzing around with the speed of light. What is more : they are bouncing off the walls of the box. And we assume that, when bouncing off the walls, they lose no energy. They exert pressure on the walls<sup>5</sup>.

We need to know a couple of things about photons to understand this pressure and analyse its effect. Photons have energy, and they have momentum.

For simplicity we are going to pretend that all the photons have the same energy. In fact it doesn't really matter for the argument, and later we will see why. It just makes it simpler. And as a matter of fact, for a box of photons in thermal equilibrium it is approximately true that they all have roughly the same energy.

Let's call the energy per photon  $\epsilon$ . I'm not using  $\epsilon$  with any deep motive. Usually  $\epsilon$  is used to denote something very small. Well, the energy of a photon is indeed very small, but that is not reason why I am using  $\epsilon$ . I use this Greek letter because it looks like  $E$ , but I want to save  $E$  for the total energy.

What about the momentum of a photon? Why do we need the momentum? Because forces are what appears in a change of momentum. And when the photons reflect on the walls, they change momentum, so they exert a force. This is exactly the same phenomenon which we saw with ordinary massive particles. If we throw a tennis ball onto a wall, the tennis

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5. It is always surprising the first time we hear about the pressure exerted by light. But as said in chapter 2, it is the principle at the root of some gadgets we can buy in stores, see the lightmill represented in figure 11 of chapter 2.

ball has some momentum. When it reflects back, if it was thrown perpendicularly to the wall, it then has the opposite momentum. There has been a change of momentum of the tennis ball. There has also been a transfer of momentum to the wall. The transfer of momentum per unit time is the *average force* on the wall – averaged over time<sup>6</sup>.

So we need to know something about the momenta of photons. Again, the momentum of a photon, I would normally denote it  $P$ . But this letter is already used for pressure. So I will denote it with the Greek letter  $\pi$ . It is a little vector, so the full vector notation will be  $\vec{\pi}$ . It has three components.

Of course the momentum of a given photon could be in any direction. If we look at the velocity or momentum distribution anywhere in the box it is isotropic. That is a fair assumption that can be justified using statistical mechanics.

So  $\vec{\pi}$  is the vector momentum of a photon. And we will use  $\pi$  without the arrow on top for its magnitude.

What is the relationship between the energy of a particle and its momentum? If we keep around the speed of light  $c$ , then the energy of a *massless particle*, i.e. a photon, is equal to the speed of light times the magnitude of the momentum.

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6. A careful analysis of the mechanics of the rebound would take into account elasticity. Obviously it is not the same thing to throw marbles or tennis balls on a wall. Even when the transfer of momentum per unit time is the same, the forces on the impacts will not be the same. But the formula  $F = m dV/dt$ , in its averaging form over time  $Fdt = mdV$ , is correct for massive particles when there is a certain elasticity, or if we introduce the concept of *average force over time*. And the analog for photons is correct too.

$$\epsilon = c\pi \tag{24}$$

That is the relationship between the energy of a massless particle and its momentum. And if we set  $c = 1$ , the energy is just the magnitude of the momentum.

Next, what about the number of photons in the box? Or better yet the number density, i.e. the number divided by the volume? Let  $\nu$  be the number of photons per unit volume.

If we recap our notations, we have

- $\epsilon$  = energy/photon
- $\vec{\pi}$  = vector momentum of a photon
- $\pi$  = magnitude of  $\vec{\pi}$
- $\nu$  = number of photons/unit volume

$\nu$  is not the density of energy. What is the density of energy in this language? It is the energy per particle times the number of particles per unit volume.

$$\rho = \epsilon\nu \tag{25}$$

Let's calculate the pressure now. To calculate it, we must have a proper theory of what the pressure is. In figure 4 is represented the wall of a hypothetical box. Think for instance of the right boundary of the box in figure 3. The gas is on the left side of the wall. It is a gas of photons.



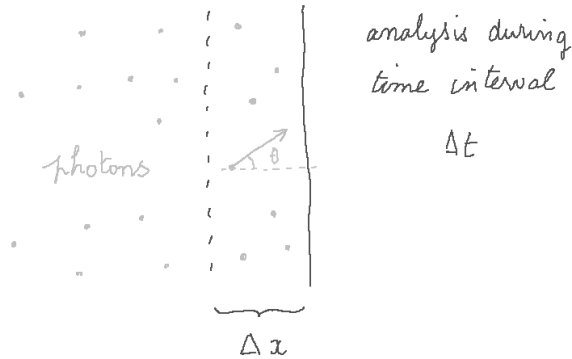


Figure 4 : Photons in a little volume of width  $\Delta x$  near the wall of the box, that will hit the wall during time interval  $\Delta t$ .

Consider a little time interval  $\Delta t$ . And consider a little volume  $dV$  of width  $\Delta x$  near the wall, figure 4. We have  $dV = A \Delta x$  where  $A$  is the area of the wall.

We are interested in how many photons hit the boundary of the box and transfer some of their momentum into it, while being reflected, in the time interval  $\Delta t$ . Generally speaking, a particle will hit the boundary during the time  $\Delta t$  if it is close enough. Notice that the speed of the particles is  $c$ , which we set equal to 1.

If a particle is moving horizontally to the left, where does it have to be in order that it will hit the boundary within time  $\Delta t$ ? It must be within  $\Delta x = \Delta t$  of the wall.

What if the particle is moving at an angle  $\theta$ , figure 4? Then it will hit the wall if  $\cos \theta > 0$  and the particle is within  $\Delta x = \Delta t \cos \theta$  of the wall.

Now consider *one particle*, moving at angle  $\theta$ , hitting the wall, and being reflected. How much momentum does it transfer to the wall? Answer : it will transfer to the wall twice the horizontal component of its momentum  $\vec{\pi}$ . That is  $2\pi \cos \theta$ . (Remember that  $\pi$  is not 3.14159... but the magnitude of the momentum of the photon.)

The reasoning is :  $\pi \cos \theta$  is the horizontal component of the momentum  $\vec{\pi}$  of the photon. When the photon is reflected by the wall, the other components of  $\vec{\pi}$  don't change, but its horizontal component changes sign. It goes from positive to negative, with the same absolute value. So its change, in absolute value, is twice  $\pi \cos \theta$ .

Now let's divide the change of momentum by  $\Delta t$ . Since, from equation (24),  $\pi = \epsilon$ , we get

$$\frac{2\epsilon \cos \theta}{\Delta t} \tag{26}$$

Why do we divide by  $\Delta t$ ? Because the force on the wall is the change of momentum per unit time<sup>7</sup>. That is the good old Newton's equation : the force on an object is the time rate of change of its momentum. And so, formula (26) is the perpendicular force exerted by each particle moving at angle  $\theta$  that hits the wall.

Now how do we find the full force exerted on the wall by all the particles moving at angle  $\theta$ ? We have to calculate how many particles, for each angle  $\theta$ , hit the wall. Equation (26) is the contribution per particle. How many particles, moving at angle  $\theta$  – or, to be more precise and to prepare

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7. We reason in absolute values.

for an integration, at an angle between  $\theta$  and  $\theta + d\theta$ , hit the wall in the time interval  $\Delta t$ ? Such particles will hit the wall if they are within  $\Delta x = \Delta t \cos \theta$  of it. Let's call the total number of such particles  $N_\theta$ . We have

$$N_\theta = \Delta x A \nu \quad (27)$$

where  $N_\theta$  is the number of particles moving at angle  $\theta$  and that will hit the wall,  $\Delta x$  is the distance within which they must be,  $A$  is the area of the wall, and  $\nu$  is the density of particles, that is of photons, in the box.

Now we can calculate the full force, due to each angle  $\theta$ , exerted on the wall. It is formula (26) for one particle, multiplied by the number of particles moving at an angle between  $\theta$  and  $\theta + d\theta$  (and for which  $\cos \theta > 0$ ).

$$F_\theta = \frac{2\epsilon \cos \theta}{\Delta t} \Delta x A \nu \quad (28)$$

But  $\Delta x = \Delta t \cos \theta$ . And the pressure  $P_\theta$ , for those particles, is  $F_\theta/A$ . So we can rewrite it

$$P_\theta = 2\epsilon \cos^2 \theta \nu \quad (29)$$

Next we shall integrate formula (29) over all possible  $\theta$ . But remember that formula (28) was derived only for angles with positive cosine. If we want, for simplicity, to integrate over all possible angles in space, with positive or negative sign, we shall work with formula (29) divided by 2.

In equation (29) we can replace  $\epsilon\nu$  by the energy density  $\rho$ . Removing the factor 2 as explained, we now have

$$P_\theta = \rho \cos^2 \theta \tag{30}$$

This is the pressure due to particles moving at an angle between  $\theta$  and  $\theta + d\theta$ , when we project their 3D motion onto the plane of figure 4.

Now what we need to do is integrate over the effect of all the different angles that particles could be moving at in 3D. Or better yet – and it is equivalent – we can ask : what is the average of  $\cos^2 \theta$  for particles moving isotropically in all directions in the three-dimensional space? We shall use a simple and elegant geometric reasoning to find the answer.

If we were looking for the average of  $\cos \theta$ , it would simply be 0. But we are interested in the average of  $\cos^2 \theta$ , which obviously is going to be a number between 0 and 1.

The particles move along unit vectors pointing in all direction in 3D. These vectors, usually denoted  $\hat{n}$ , have components  $n_x$ ,  $n_y$ , and  $n_z$ . What we are interested in is the average of  $n_x^2$ . The trick is to note that the following relationship always holds

$$n_x^2 + n_y^2 + n_z^2 = 1$$

So the average of the left-hand side must be 1. Therefore, since the  $x$ -axis, the  $y$ -axis and the  $z$ -axis play exactly the same role, the average of  $n_x^2$  is  $1/3$ . If there were four directions of space it would be  $1/4$ . If there were two directions of space it would be  $1/2$ . If there was only one direction of space it would be 1. But in three dimensions it is  $1/3$ .

In other words, turning back to equation (30), we found that the total pressure exerted on the walls by the photons is

$$P = \frac{1}{3} \rho \quad (31)$$

That is the derivation of the equation of state for radiation.

Incidentally, did we really restrict the problem when we assumed that all the particles had the same energy  $\epsilon$ ? No. Equation (31) could be thought of as the contribution from particles of a given energy. Then add them all up. It doesn't matter. We get the same answer.

Question : do we get the same answer if we consider black absorbing walls instead of reflection walls? If we are in thermal equilibrium, with similar calculations we can show that the answer is yes.

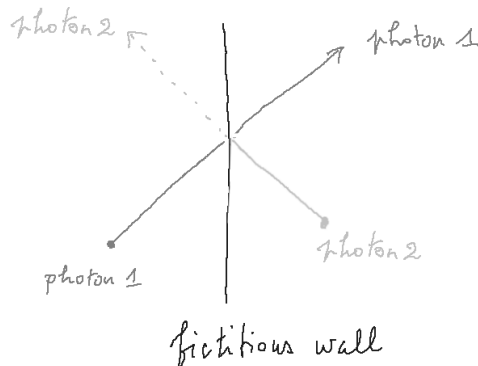


Figure 5 : Any fictitious wall is equivalent to a real one.

But a better question is : what walls are we talking about ?

There is no wall out there in space. Photons are not reflecting off a wall, figure 4. But if we imagine a fictitious wall anywhere in space, what will happen in its vicinity is exactly equivalent to it being real : on average, for any photon going through it instead of being reflected, there will be another photon crossing the wall in the symmetric direction, figure 5.

So on the average the fictitious wall of the box we are thinking about, really does behave as though it is really there. What we did is the same as plunging a flat piece of wood, real or fictitious, into a liquid to construct a reasoning to calculate the pressure from considerations of forces exerted on the piece of wood.

In other words, it really doesn't matter what our model for the origin of pressure is. Without walls it is the same. Radiation pressure is  $1/3$  the energy density.

The radiation in the universe is almost all the cosmic microwave background, and it is in thermal equilibrium.

So we have tied up the last loose end. Now we understand, in the matter-dominated case,

- a)  $\rho = \rho_0/a^3$
- b)  $a(t) \sim t^{2/3}$
- c)  $w = 0$

and in the radiation-dominated case,

- d)  $\rho = \rho_0/a^4$
- e)  $a(t) \sim t^{1/2}$
- f)  $w = 1/3$

We are ready to move on to new kinds of equations of state.

Question : Would the results be different if we used quantum mechanics ?

Answer : No. The equation of state stays essentially the same  $P = w\rho$ . The differences are really small. They would come from what happens in collisions between photons. But the cross section of photons – which governs the probability of interaction between them and of possible scattering – is too tiny for that to be significant.

Actually, to give a more thorough answer, it depends on the temperature and the density. If the temperature and the density are high enough, so that photons can collide frequently and can produce electron-positron pairs, then the equation of state will change. But the temperature would have to be exceedingly high, way beyond what we are implicitly considering here.

### **Negative pressure**

The reader should recall the example of a physical system where there is negative pressure. Let's review it briefly. For simplicity it is in one dimension. Consider a box, with two walls, i.e. two end points. If the box is filled with particles whizzing around and hitting the walls, there will be positive pressure on the walls.

Now imagine the two walls, not with particles inside creating positive pressure, but tied by a spring, as in figure 6. With an adequate spring, it doesn't push the walls out, it pulls them in. That is negative pressure. It has of course another name : it is called tension. The tension of the spring is effectively a negative pressure.



Figure 6 : Example of negative pressure, with a spring tending to pull the walls together.

This is the same for a box in the universe. For positive pressure when we increase the size of the box, we do some work on the wall of the box. And the energy decreases inside. But tension, when we pull against it, we increase the energy inside.

So if for some reason or other we had negative pressure and, let's say, positive energy, then  $w$  would be negative. That is a possibility.

If it was not absolutely central in cosmology, we would not be talking about it. But it is a central fact that the pressure  $P$ , in the equation of state, can be negative, *even if energy density is positive*.

In a box, pressure will be positive if we have a bunch of particles moving around which don't interact with each other



very much and mainly just bounce off the wall. On the other hand, if the particles are attracting each other, they are pulling themselves together. And if they also attract the wall, they will pull the walls in. So there are certainly circumstances where pressure can be negative. And, as we said, it corresponds to tension.

We are going to talk about an example called *vacuum energy* where the pressure typically is negative.

### **Vacuum energy**

Vacuum energy is a special kind of energy in the universe. Its existence is a consequence of quantum field theory. But we don't need to know where it comes from to describe it. It is just an energy that we assign to empty space.

In our bookkeeping of all the things making up the universe, we can conjecture that empty space with nothing in it in fact has energy.

Now, we know where the vacuum energy comes from in quantum mechanics. It comes from zero point energy of fluctuation, that is zero point energy of harmonic oscillators which represent the quanta of the field, see chapter 10 of volume 2 in the collection *The Theoretical Minimum*, on quantum mechanics. So we know where it comes from. But whatever its origin, it is energy that is simply there in empty space.

It is as if this page had a uniform energy density on it. And nothing we could do, short of putting in more material and so forth, would change the energy of that page. It is just a fixed thing that is there.

Now let's consider an empty box, figure 7. It could be a box with fictitious walls, or it could be a box with real walls. How much energy – vacuum energy in this case – is there in the box? The answer is : whatever the vacuum energy density is, multiplied by the volume of a box.



Figure 7 : Empty box.

Vacuum energy has the special property that its density is a constant. By that we mean that the *density* of vacuum energy inside the box does not change when you change the size of the box. It is just a characteristic of empty space in our universe. And, as long as the box only contains empty space, the energy density inside the box is fixed.

The reader may have heard of the Casimir<sup>8</sup> force, and object that even with empty space inside the box there is an energy density varying when we move the walls. But

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8. Hendrik B. G. Casimir (1909 - 2000), Dutch physicist.

the Casimir force is only important when the box has real walls, and these walls are really very close together. For our purposes, we are just talking about energy density which is there, no matter what we do.

The first characteristic we should note about the vacuum energy density is that, unlike energy density coming from matter or from radiation contained in the box and which is diluted if we increase the volume of the box, the vacuum energy density does not change when we change the volume of the box for instance by expanding its walls.

That is what may be counterintuitive about vacuum energy : it is there, but it cannot be compressed or diluted. It is a *quality* of space, more than something behaving like a quantity, which we could compress or dilute. Yet we can compute the *quantity* of vacuum energy in a volume by multiplying the density by the volume.

The vacuum energy density has a name. It is called  $\rho_0$ . The naught stands for vacuum. Now we shall introduce another variable, named  $\Lambda$ , which is just  $\rho_0$  multiplied by a constant. By definition

$$\Lambda = \frac{8\pi G}{3} \rho_0 \quad (32)$$

The vacuum energy density is  $\rho_0$ .  $8 \pi G$  and  $3$  are just numerical constants. So  $\Lambda$  is not a fundamentally different concept than the vacuum energy density. But  $\Lambda$  has its own name : it is the *cosmological constant*.

The relation between the left side and the right side of equation (32) is trivial. It is just the definition of  $\Lambda$  as a multiple of  $\rho_0$ . However it is useful to introduce  $\Lambda$ , in particular in

the Friedmann equation. We met the Friedmann equation, in chapter 2, in the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \dots \quad (33)$$

The other term represented by three dots can be some constant over  $a^2$  for instance. But there is always the factor  $8\pi G/3$  in front of  $\rho$ . So when, in equation (33),  $\rho$  is  $\rho_0$ , the first term on the right simply becomes the cosmological constant  $\Lambda$ . In other words  $\Lambda$  appears nicely in Friedmann equation, while the vacuum energy density  $\rho_0$  appears less nicely. Nevertheless let's think of  $\rho$  as energy density.

Incidentally, for vacuum energy we know immediately what the relation is between the vacuum energy density and the scale factor : there is no relation ! Remember that figuring out, in various models, the relation between  $\rho$  and  $a$ , in order to get a workable differential equation in  $a$ , and find out the history of the scale factor, has been our main concern in all the lessons so far. For vacuum, as stressed, no matter how big or small we make the box, the vacuum energy density is always the same.

$\rho_0$  is the universal energy density in the vacuum. And it doesn't change when we change the size of the universe. We are talking about the energy *density*, not the energy. So we already know the answer to how  $\rho$ , the energy density in the vacuum, varies. It does not.

But let's ask, for fun, about its equation of state.

## Equation of state for the vacuum

What kind of equation of state does the vacuum correspond to?

The vacuum does correspond to an equation of state. To figure it out we go back to equations (13) and (16), which stated that  $dE = \rho dV + d\rho V$  and that it is also equal to  $-w\rho dV$ .

Since  $\rho$  does not change, but stays  $\rho_0$ , in equation (13) we have  $d\rho = 0$ . So we can rewrite it

$$dE = \rho_0 dV \tag{34}$$

And equation (16) now becomes

$$\rho_0 dV = -w\rho_0 dV \tag{35}$$

So – skipping three pages of calculations :- ) – we reach the conclusion that

$$w = -1 \tag{36}$$

When we read, in various places, that astronomers are measuring  $w$  and are discovering that  $w$  is close to  $-1$ , that is what they are talking about. They are talking about vacuum energy.

The closer the experimental evidence is that  $w = -1$ , the more the energy density of the universe is like vacuum energy. It doesn't dilute when we expand space. It doesn't dilute because it is a property of empty space to begin with.

So this is the vacuum energy. It can be positive or negative. In either case the pressure and the energy density have opposite signs. That is the meaning of  $w = -1$ . If the energy density of the vacuum is positive, the pressure is negative. If the energy density is negative, the pressure is positive. It is a characteristic of vacuum energy. It is not intuitive. But after a while, it becomes familiar. It is not all that crazy.

When we think about energy and pressure in terms of particles, and the usual phenomena we are used to thinking about causing pressure, first of all negative pressure may seem a little odd. Especially odd is this fact of the energy density and the pressure having opposite signs. We tried to build some intuition about it with the help of springs, figure 6, and particles attracting each other. Anyhow, what it comes down to is the little derivation of  $w$  shown above in equations (34) to (36).

So, in vacuum, pressure is the opposite of energy density. It is the equation of state for an empty universe, assuming that it is governed by vacuum energy,

$$P = -\rho_0 \tag{37}$$

Now what is the value of  $\rho_0$ ? That is something we don't know how to compute. There are too many contributions to it. They come from all sorts of quantum fields that we may not have discovered yet. Some come from high energies, some come from low energies. One would have to have a pretty exact theory of all the quantum fields in nature to be able to compute the value of  $\rho_0$ . And we haven't got the vaguest idea of why it has the numerical value it has.

We will talk about the numerical value of  $\rho_0$  later. Let's just say for now that it is extremely small. We will be mostly concerned, however, with the implications of it. The energy density of the vacuum is one form of energy, and it competes with the other forms of energy. But let's first study the special case where the only energy density in the universe is vacuum energy.

Just like we studied the pure matter-dominated case, then the pure radiation-dominated case, then we mixed the two of them, and we said radiation dominated early, matter dominated afterwards, now we are going first to study the pure vacuum energy case. Later we will incorporate it in a more complex model.

So we want to isolate out just what pure vacuum energy density would do. Let's go back to the equations governing the expansion of the universe, and see how vacuum energy would influence things.

There are six cases. The cosmological constant  $\Lambda$ , which is proportional to the energy density, can be positive or negative. That is two cases. Of course there is actually an infinite number of cases for the numerical value of  $\Lambda$ . We are talking about the two cases of *sign* for the cosmological constant. And there is in each case, the different possible values of the curvature parameter  $k$ .

Remember that  $k$  is the curvature of space. It can be  $+1$ ,  $0$ , or  $-1$ . It is  $+1$  for spherical space,  $0$  for flat space, and  $-1$  for hyperbolic space. That is three cases for  $k$ . And altogether that makes six cases.

And what are the equations? They are the good old Friedmann equation in the various cases. Let's write it down

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 - \frac{k}{a^2} \quad (38)$$

Remember that  $\dot{a}/a$  is also called the Hubble constant. And on the right-hand side the energy density  $\rho$ , as we saw, does change, so it is only  $\rho_0$ . Finally the last term on the right-hand side is related to the curvature of the space as we showed in the previous chapter.

Equation (38) is the equation we would like to solve. Before we do so, let's just take advantage now of our definition that  $8\pi G/3$  times  $\rho_0$  is called  $\Lambda$  (read *lambda*).

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{k}{a^2} \quad (39)$$

$\Lambda$  was introduced to get rid of the nasty  $8\pi G/3$ . It is called the cosmological constant. It was introduced by Einstein who later rejected it.

Lambda is also called *dark energy*. It is the thing that the newspapers call dark energy – dark because it doesn't glow.

As we said, we can have lambda positive, or negative. We can even have lambda equal 0 incidentally. If  $\Lambda = 0$  we have already done the calculations, see equation (21) of chapter 2. In that case  $k$  can only be negative. It corresponds to a hyperbolic space, and we saw that its radius expands with a constant velocity  $\dot{a}$ .



In fact, even with lambda non zero, we can see that there are actually fewer than six cases, because some don't make sense. For instance, suppose that  $\Lambda$  is negative and  $k$  is positive. Then equation (39) equates a square of the left-hand side to something negative on the right-hand side. So it cannot make physical sense<sup>9</sup> because  $a$  and its derivative are real functions of time.

There is a number of other cases that don't have any solutions, or at least solutions that don't make sense.

### De Sitter space with flat geometry

Let us study one case which does have a solution. By far the simplest is  $\Lambda > 0$  and  $k = 0$ . Then we will come back to the other cases. When the curvature parameter  $k = 0$ , we have a flat universe.

So the universe is a three-dimensional space flat and empty, with positive vacuum energy density. It may be somewhat counterintuitive but, despite its emptiness, its geometry evolves. The scale factor satisfies

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda \tag{40}$$

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9. Remember how we use equations in general in mathematics or physics : we are looking for an unknown  $x$ . Some reasonings lead to figure out that  $x$  must satisfy such and such equation. We solve the equation. If it has several solutions, that doesn't mean that  $x$  can be any of them. It only means that  $x$  cannot be anything else than one of them.

Let's solve this equation. We take the square root, and write

$$\dot{a} = \sqrt{\Lambda} a \quad (41)$$

or

$$\frac{da}{dt} = \sqrt{\Lambda} a \quad (42)$$

That is the equation that says that the time rate of change of something is proportional to that something. What is the solution of such an equation? Exponential growth.

Notice, by the way, that if  $\Lambda$  were negative we would be having a problem in equation (41). It would make no sense.  $k = 0$  and  $\Lambda < 0$  is another example of meaningless case.

But for  $\Lambda > 0$  and  $k = 0$ , there is a solution.

$$a(t) = c e^{\sqrt{\Lambda} t} \quad (43)$$

We can put any constant  $c$  we like in front. It doesn't matter. It doesn't change the geometry. It only changes the unit length with which  $a$  is measured.

Equation (43) is an interesting case : the universe exponentially expands. That is a consequence of vacuum energy. We see that

*in the case of a flat empty universe with positive vacuum energy, the scale factor  $a$  grows exponentially with time.*

Let's calculate the Hubble constant. But there is no calculation to do! Equation (40) readily gives us the Hubble constant. It is

$$H = \sqrt{\Lambda} \quad (44)$$

In this case, the Hubble constant – which is always independent of space in a homogeneous universe – is also independent of time. And we can also rewrite equation (43) as

$$a(t) = c e^{Ht} \quad (45)$$

This is called the de Sitter space<sup>10</sup>. De Sitter discovered this solution of Einstein's equations, with a cosmological constant, in 1917, that is only two years after Einstein published his theory of general relativity. Equation (45) is actually one version of de Sitter space with exponential expansion.

It can be proved that this is the unique geometry such that the Hubble constant  $H$  is constant in time. There remains some ambiguity about the geometry though. If we want to use a technical set of words, this geometry is not geodesically complete. There are trajectories back into the past which go to the infinite past in a finite proper time. That means that some of the geometry is missing. But we will take up this point later. It is a problematic question whether there was a Big Bang in this kind of space or not.

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10. named after Willem de Sitter (1872 - 1934), Dutch physicist and astronomer.

Anyhow this kind of space doesn't exist by itself. There is no reason why we shouldn't put back other kinds of things into it. And we will indeed consider other kinds of material making up the universe – that is, appearing on the right-hand side of equation (40) –, not only this strange vacuum containing energy. We will add matter, radiation, energy densities which change with  $a$ .

So in equation (38) let's put  $\rho_0$ , that is  $\Lambda$  which absorbs the constant in front of  $\rho_0$ . Then we add some radiation, that is  $c_R/a^4$ ; some matter, that is  $c_M/a^3$ ; and finally the term involving curvature,  $-k/a^2$ , which can be seen as "energy density due to curvature".

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \frac{c_R}{a^4} + \frac{c_M}{a^3} - \frac{k}{a^2} \quad (46)$$

This is our complete Friedmann equation relating the square of the Hubble constant, on the left-hand side, to the various energy densities in the universe, on the right-hand side. The first term, on the right-hand side, is the cosmological constant. It is the energy density  $\rho_0$  of the vacuum, multiplied by  $8\pi G/3$ , that makes it simply  $\Lambda$ . It doesn't depend on  $a$ . The second term is proportional to the radiation density. We saw that it varies like the inverse of  $a$  to the 4th power. The third term is proportional to the matter density. It varies like the inverse of  $a$  cube. And the last term is the curvature energy density. It varies like one over  $a$  squared.

When  $a$  is small, the dominant term in equation (46) is  $c_R/a^4$ . On the other hand, when  $a$  gets big enough,  $\Lambda$  will dominate, and the universe will exponentially expand.

So in very early times the vacuum energy was not important. In very late times, it will dominate everything else. And there are transition periods in between : from radiation-dominated to matter-dominated, and then from matter-dominated to exponential expansion.

When we make our observations, the universe appears now to be in transition from matter-dominated to exponential expansion. The term in  $1/a^4$  has become negligible a long time ago. And the two terms,  $\Lambda$  and  $c_M/a^3$  are competing with each other. At present,  $\Lambda$  is slightly bigger than  $c_M/a^3$ . So we are not yet seeing genuine exponential expansion.

It is the meaning of all the curves we have drawn representing the evolution of  $a$  over time, see figures (10) and (12) of chapter 2. They show something which looks very much, in early times, like a radiation-dominated universe, followed by a matter-dominated universe. This first transition took place about 50 000 years after the Big Bang, which is itself placed approximately 13.8 billion years ago. But over the last 2 or 3 billion years, we begin to see a deviation from a growth like  $t^{2/3}$  which was characteristic of a matter-dominated universe. The deviation is pointing upward like the exponential  $e^{Ht}$  with a positive  $H$ .

Why is it called an accelerating expansion ? For the simple reason that if  $a$  increases exponentially and we calculate the acceleration – that means the second time derivative – it is also increasing exponentially. The derivative of an exponential is another exponential. The second derivative is yet another exponential. And so, the universe is not only expanding, but is it expanding in an exponential way, and

that is true in an accelerated way, etc.

It could be accelerated without being exponential, incidentally, for instance if  $\ddot{a}$  was simply positive.

So what is the truth? The observations at the present time confirm acceleration. And the more precision we get, the more it looks like it is beginning to exponentially accelerate.

### Questions / answers session

Question : You said that having no vacuum energy is weirder than having some vacuum energy. But how do you explain negative vacuum energy?

Answer : If you calculate the vacuum energy of a quantum field, it will be positive for bosons and negative for fermions. That is just a fact of mathematics, that the vacuum energy for bosons is  $\bar{h}\omega/2$  for each fluctuating mode, and that it is  $-\bar{h}\omega/2$  for each fermionic mode.

But we are not going to try to answer the question of where the vacuum energy comes from, what it is due to.

As I said indeed, what is much weirder than having vacuum energy is having no vacuum energy. There is no known theory, that is in anyway consistent with the world as we know it, which would predict zero vacuum energy.

So when people talk about the mysterious dark energy, what they should be saying is it is very mysterious that

there is so little of it.

We can discuss in what sense it is numerically very small. It must be numerically very small in some sense, since it took so long to discover it. If it was big enough to cause an exponential expansion that we could see in the room, of course it would blow everything apart. It would be a disaster for us.

But, even without talking about such a big effect directly perceptible by us, if it had any appreciable size, then we would have discovered it long ago. Well, we did discover it<sup>11</sup>. But it was very hard to discover. It took enormously big telescopes, in effect seeing to the end of the universe. And that is an indication that in some sense it is a very small number.

Q. : We hear that the universe may eventually be torn apart by a Big Rip, itself a consequence of exponential expansion. Can you say more about this model?

A. : I never follow very much what is published about the Big Rip. It seems to me one of those ideas which the press like more than any physicist I know.

As I understand it, the Big Rip is what would happen if  $w$  was even more negative than  $-1$ . And there is no known sensible theory where  $w$  was more negative than  $-1$ .

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11. Theoretical speculations about the cosmological constant date back from Einstein's theory of general relativity. But the first observational evidence of dark energy and exponential expansion were obtained in 1998.

Incidentally, observations seem to be zeroing in on  $w = -1$ . It is measured now somewhere between  $-1.1$  and  $-0.9$ , with diminishing error bars. I think it will never be a high precision number – like for instance the mass of an electron whose measure is  $9.10938356 \times 10^{-31}$  kilograms – but they can probably narrow it even more.

Q. : If we had exact supersymmetry, couldn't the vacuum energy density be zero ?

A. : It could. And the reason is because every fermion comes along with a boson. So they cancel each other exactly.

When I said that there is no theory that agrees with everything we know about nature, I had that in mind. We know for a fact that the numbers of fermions and bosons in the universe don't exactly match. Therefore supersymmetry is not totally consistent with observations.

Q. : The actual vacuum energy density in the universe is extremely close to zero, although there is a tiny discrepancy. What is it ?

A. : The discrepancy is of the order of  $10^{-123}$ . That is a very small number ! Let me explain why the incredible smallness of this number is surprising. We have to talk about the numbers produced by nature that we call *constants of nature*.

Most of them are not terribly fundamental. For example the mass of the electron mentioned above is thought to be



the consequence of more complicated stuff.

The really fundamental constants of nature are

1.  $c$ , the speed of light,
2.  $h$ , Planck's constant,
3.  $g$  (or  $G$ ), Newton's gravitational constant.

Why are they fundamental? Because there is a sense in which they are about universal things. Let's discuss this a little bit.

What is universal about  $c$ ? It is the fact that nothing in the world can go faster than  $c$ . No signal can go faster than the speed of light. It really does have a universality to it. It is not conditional on saying something like : we are going to be using this or that technique to send messages. It is true no matter what. You simply can't get past it.

What about Planck's constant? It is also universal. It has to do with the uncertainty principle. Irrespective of which object we are talking about, be it a bowling ball or an electron, uncertainty in position times uncertainty in momentum is always greater than or equal to half Planck's constant <sup>12</sup>.

$$\Delta X \times \Delta P \geq \frac{\hbar}{2} \quad (47)$$

So it has a universal aspect to it.

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12.  $\hbar = h/2\pi$ . Somewhat nonchalantly, both are called Planck's constant. However one is called "h" and the other "h bar".

Newton constant is also quite universal. Think of the law of gravity. All objects, without exception, exert forces between them due to gravitation. They are equal to the product of their masses times Newton's constant divided by the square of the distance between them.

$$F = \frac{mMg}{d^2} \quad (48)$$

where  $m$  is the mass of one of the object,  $M$  is the mass of the other,  $d$  is the distance between them, and  $g$  is Newton's constant.

It is the use of the word *all*, in the three cases, which says that there is something deep and fundamental about these constants.

There are other constants that we sometimes talk about. It is probably true that all protons and electrons have the same ratio of mass of 1836. So we could say that it is a constant of nature. But there are zillions of particles, or at least lots of different kinds of particles. And ratios of particle masses are not in any special way universal.

So we tend to think of  $g$ ,  $\bar{h}$  and  $c$  as very fundamental.

Out of those constants we can make an energy density. First, we make a unit of energy. It is called the Planck energy. The Planck energy corresponds to the energy of a mass of 4.341 micrograms<sup>13</sup>. In other words it is a microscopic mass, but a "macroscopic microscopic" mass, in the sense that it is much bigger than the mass of elementary

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13. This is the so-called *reduced* Planck energy.

particles. It is the mass of a grain of fine sand. If it were to annihilate, the energy that would be released would be the Planck energy or the Planck mass. It is about the same as the conventional energy in a tank of gasoline. It is a lot of energy.

Then there is the Planck length. It is very small<sup>14</sup>. Finally there is the Planck time<sup>15</sup>. They are the units of mass, length and time that we can make out of  $g$ ,  $\hbar$  and  $c$ .

Or another way of saying it is :

*the Planck units of length, mass and time are such that  $g$ ,  $\hbar$  and  $c$  are all equal to 1.*

From Planck length we can define a unit of volume. It is  $4.22 \times 10^{-105}$  cubic meter.

If we have a unit of mass, we have a unit of energy. It is one Planck energy.

Then, from the unit of energy and the unit of volume, we build a unit of energy density. It is one Planck mass per cubic Planck length. That is the natural, universal unit of energy density. By the way, how big is that? The Planck volume is tiny, and the Planck mass is pretty big, so it is a huge energy density, vastly bigger than anything we ever experience in the ordinary world.

On the other hand, what do we mean by huge? We mean huge by comparison with us, ordinary creatures. The Planck

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14. Planck length =  $1.616 \times 10^{-35}$  meter.

15. Planck time =  $5.391 \times 10^{-44}$  second

energy density, however, is the only unit of energy density that occurs naturally in basic physics.

How big is the actual vacuum energy density of our universe? It is nowhere near as big as that natural energy density. It is about 123 orders of magnitude smaller. Let's explain why this incredible small ratio is strange.

From somewhere unknown to us there is a tiny energy density of the vacuum, which is 123 orders of magnitude smaller than what we might guess. Nobody knows how to calculate it.

What we might have guessed off hand is that it would be approximately one in natural units, or something of the same order of magnitude. Well, to be honest, we would never have guessed that, because we would not be there to guess it if it were true.

But if you were to take your random guess about what the set of laws of nature would produce, it would be 123 orders of magnitude bigger than what we see. That is what is surprising about the exceeding smallness of the vacuum energy density.

In this course, we won't ask where this vacuum energy comes from. But we should take note of what is mysterious about it. What is mysterious about it is not that it is there; it is that it is almost not there. It is the lack of it which is the mysterious fact.

Q. : You say that what is surprising about vacuum energy

is that there is so little of it. But when astronomers discovered it, weren't they surprised that it was even there at all?

A. : Yes, but that was more psychological than anything else. You know, at some time in the history of astronomy, it was possible to detect it, let's say, at the level of  $10^{-100}$ .  $10^{-100}$  is a rather big energy density, incidentally. I think it is probably much too big. Anyway sometime in the history of astronomy,  $10^{-100}$  could be discovered, but not  $10^{-101}$ , it was too small.

Astronomers did not discover it at the level of  $10^{-100}$ . It seemed to be zero at that level. So they pushed ahead, devised more precise techniques, and looked for it at the level of  $10^{-101}$ . Still zero.  $10^{-102}$ ,  $10^{-103}$ ...  $10^{-121}$ ,  $10^{-122}$ , still zero. When you measure like that, with incredibly growing precision, and always get zero, you get the feeling that maybe this thing really is zero for reasons that we don't know – because as I said no sensible theory predicts it to be zero.

This was an attitude that had affected almost everybody in physics, astronomy and astrophysics communities. Einstein himself didn't believe in lambda, for a different reason though. But the fact that it was so small and each time another decimal was added to the knowledge of it, it was still zero, simply got people convinced that it might be actually zero. There must be some reason.

Now the logic was crazy logic. The cosmological constant seemed to be exactly zero. It must be a consequence of the right theory of nature. If we had the right theory of nature, we would prove it. And of course everybody knows the right

theory of nature is string theory<sup>16</sup>. Therefore it must be a consequence of string theory. So they thought : ah, we just explained it !

That mental process was really there : now that we have a theory of gravity and quantum mechanics, and we know that the cosmological constant is zero, our theory must predict it. If it predicts it, we win, we are successful.

But the best-laid plans of mice and men often go awry. And it didn't turn out that way.

Q. : Can this vacuum energy density value change over time ?

A. : Your question is related to measuring  $w$ <sup>17</sup>. Obviously you cannot discover whether  $\rho_0$ , or equivalently  $\Lambda$ , will have changed after a trillion years. There is no way to do that. But you can try to discover whether over the last billion or 2 billion years it might have changed by a small amount. That is equivalent to measuring  $w$  more precisely.

At present  $w$  is measured to be  $-1$  with a precision of about 10%. That is evidence that at least over the relatively short term – a few billion years –  $\Lambda$  hasn't changed much. It hasn't changed by more than a few percent. I think we will never be able to nail it completely.

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16. Professor L. Susskind is one of the fathers of string theory.

17. If  $\rho$  is fixed, then  $w = -1$ , see equations (34) to (36). And if  $\rho$  changed over time,  $w$  would not be  $-1$ .

Q. : Does this analysis depend on the number of dimensions in the universe? In other words, would we see things differently if we lived for example in four spatial dimensions?

A. : No, not really. They are pretty similar. They don't depend much on dimensions. The details would be different; the general pattern would be the same.

### De Sitter space with spherical geometry

We have studied above the evolution of a flat de Sitter space, that is an empty universe with positive vacuum energy density and flat curvature  $k = 0$ .

Let us now do another case just for fun. Let's do the case  $\Lambda > 0$ , but instead of  $k = 0$ , the curvature parameter is  $k = +1$ .

This is a positively curved, i.e. spherical, universe. And we still look at the case of a positive cosmological constant. It is an interesting case.

First of all, let us see how we can intuit the rough properties of it. And then we will look at the exact solution. We have

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda - \frac{1}{a^2} \quad (49)$$

This is our equation with  $k = +1$ . Let's see if we can make some sense out of it.

One technique to make sense out of it – which is useful in many situations when we reach an equation that we have to solve – is to try to see if it has the same structure as an equation we are already familiar with because it governs a physical system we know well.

First of all, we multiply both sides by  $a^2$ . Transposing the term with  $\Lambda$  to the left side, we get

$$\dot{a}^2 - \Lambda a^2 = -1 \quad (50)$$

Now, think of  $a$  as the coordinate of a particle. It has to be positive, because the scale factor is always positive, but let's ignore that for a moment and just think of it as the coordinate of a particle on a line. We usually call such a coordinate  $x$ , but here we call it  $a$ .

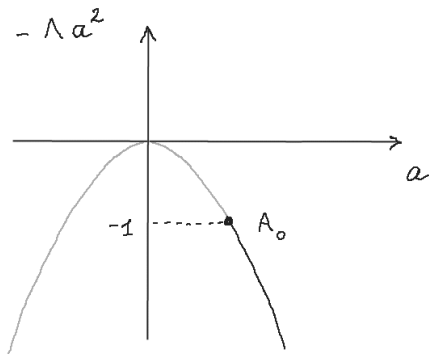


Figure 8 : Motion of particle with total energy, kinetic plus potential, equal to  $-1$ . It moves on the parabola on the right of  $A_0$ .

The first term  $\dot{a}^2$  would then be proportional to the kinetic energy of the particle. And the second term,  $-\Lambda a^2$ , on



the left, could be like a potential energy. It would be of a slightly unusual kind because of the negative sign. Instead of being a parabola opening upward, like for a mass attached to a spring or a marble rolling in a bowl, it would be an upside down parabola, see figure 8.

Equation (50) says that the total energy, kinetic plus potential, of the fictitious particle is equal to  $-1$ . How would that particle move? Suppose we released the particle, with no speed, at point  $A_0$ , where the potential energy is  $-1$ . It would then start to accelerate to the right and, at the same time, have its potential energy increase in the negative values. It would roll down on the parabola.

We could also launch the particle with some positive kinetic at the start, that would mean releasing it farther down the parabola with some initial speed. If the initial speed was to the left, the particle would go up to  $A_0$  and then fall back. That would be a kind of bounce. Or, the initial speed could be to the right, then the particle would speed up right away falling along the parabola.

So either  $a$  gets bigger and bigger from some point to the right of  $A_0$ . Or  $a$  starts big, shrinks, comes to rest at the abscissa of  $A_0$ , and then goes back down. That is what equation (50) is describing.

If we solve equation (50), basically we get the motion of a particle that comes up the hill and back down the hill.

What does  $a$  look like as a function of time? We can plot it against time, figure 9.

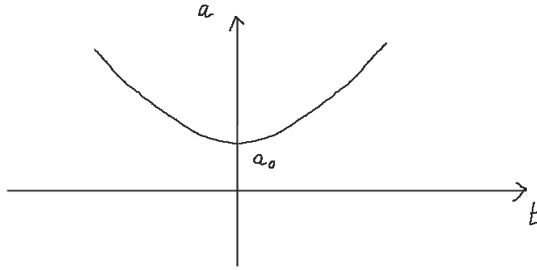


Figure 9 : Scale factor  $a$  as a function of time.

$a$  starts out big in the past at negative time. It shrinks to  $a_0$  and goes back up. In other words, this is some new kind of cosmology where the universe shrinks, reaches some size, bounces, and comes back out.

Can we solve equation (50) exactly? Yes. It is not hard to solve exactly. It is left as an exercise to the reader.

**Exercise 1 : Solve equation (50).**

The reader can check that the solution is

$$a = \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda} t) \tag{51}$$

That is the exact solution. Let us remember what hyperbolic cosine is.

$$\cosh(\sqrt{\Lambda} t) = \frac{1}{2} \left[ e^{\sqrt{\Lambda} t} + e^{-\sqrt{\Lambda} t} \right] \tag{52}$$

So  $a$  is this divided by  $\sqrt{\Lambda}$ .

It is a symmetric function of time. That is, it takes the same value at time  $t$  and at time  $-t$ . When  $t$  is big, the term  $e^{-\sqrt{\Lambda} t}$ , in equation (52), becomes negligible. But  $e^{\sqrt{\Lambda} t}$  exponentially increases. So in figure 9, the scale parameter exponentially increases into the future, and also into the past.

We can draw this universe as it evolves with time. Remember that it is a spherical universe. To draw a visually meaningful picture we have to consider a one-dimensional positively curved universe, that is a 1-sphere, in other words a circle, and plot it as it evolves with time, figure 10.

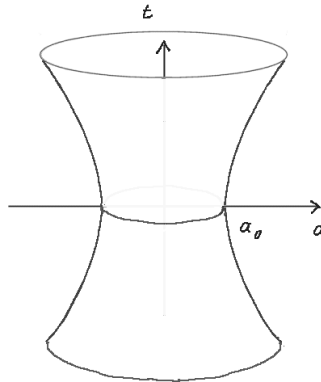


Figure 10 : Spherical universe expanding with time. At time  $t = 0$  the 1-sphere universe is the smallest circle of radius  $a_0$ .

At time  $t = 0$ , the 1-sphere universe has its minimum radius  $a_0$ . Then as time goes forward into the future (upper part of figure 10) the universe expands. The scale factor

increases exponentially. The same is true as time goes into the past (bottom part of the figure).

This is a strange kind of universe which exponentially increases in the future.

Remember that the flat case also exponentially increased in the future. The flat case had only  $e^{\sqrt{\Lambda} t}$  in the solution for  $a(t)$ , see equation (43). It did not have the term  $e^{-\sqrt{\Lambda} t}$ . But at late times, they basically both just exponentially expand, and they look very similar.

The whole geometry, from negative infinite past to positive infinite future, is usually described as a *bounce*.

Now we don't believe that the lower half of figure 10 means anything. Nevertheless the whole thing is the mathematical structure of the universe if it only contained positive vacuum energy, and its curvature parameter was  $k = +1$ . The universe would be a bounce.

This is called also a de Sitter space.

Now strangely the flat case and this spherical case are really secretly the same geometry. We will try to see why when we get to it. They are really not different, but that will take some time to explore. They look very different but they are not.

Let's finish this lesson with another case. Let's try the case with a negative cosmological constant, that is a negative vacuum energy, and see what we can learn.

## Universe with negative vacuum energy

We consider the case where  $\Lambda < 0$ . To make it explicit, we can write Friedmann equation as follows

$$\left(\frac{\dot{a}}{a}\right)^2 = -|\Lambda| - \frac{k}{a^2} \quad (53)$$

And in fact to make life simple, let's take the case where  $|\Lambda| = 1$ . It is true in some units, therefore we don't restrict the scope of our analysis doing so. Equation (53) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = -1 - \frac{k}{a^2} \quad (54)$$

We readily see again, as we mentioned before, that this equation is impossible if  $k = +1$  or  $0$ . So we consider only the case where  $k = -1$ . Our equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = -1 + \frac{1}{a^2} \quad (55)$$

Let's use the same technique we used before : let's look at it as the equation of a simple mechanical system. What mechanical system this could correspond to? With a positive cosmological constant and positive curvature we interpreted Friedmann equation as that of a particle rolling along a parabola opening downward. Let's see what we can do now.

As before, we multiply by  $a^2$ , and after some rearranging we get

$$\dot{a}^2 + a^2 = 1 \quad (56)$$

If we had kept the  $\Lambda$  around, it would just multiply  $a^2$ . But let's work with equation (56). Now what kind of system are we talking about ?

If  $\dot{a}^2$  is kinetic energy and  $a^2$  is potential energy, it says that the total energy of a particle at position  $a$  is constant and equal to 1.

What kind of system has a potential energy which increases quadratically with displacement ? The harmonic oscillator. So equation (56) is actually the energy conservation equation of a harmonic oscillator, with in this case a unit spring constant.

Again let's represent the potential energy as a function of  $a$ , figure 11.

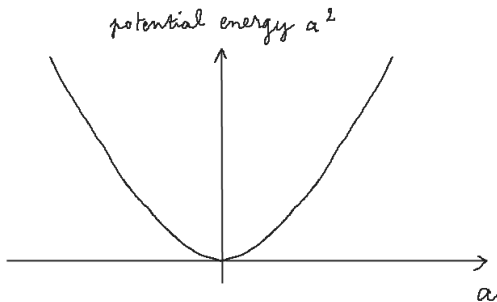


Figure 11 : Potential energy of a harmonic oscillator. A hyperbolic universe with negative cosmological constant behaves like a harmonic oscillator : the trajectory of its scale factor is the positive half of one sinusoidal period.

When the position of the particle is at  $a = 0$ , its potential energy is 0, and its kinetic energy has the maximum value

of 1. The particle will climb along the parabola, to the right or to the left, up to a potential energy of 1, come to rest (i.e. kinetic energy = 0), and roll down again.

Of course, since  $a$  is in reality the scale factor of the universe, only the right part of figure 11 is meaningful. We can forget about the part on the left of the vertical axis.

The position  $a = 0$  corresponds to the time zero of the Big Bang. Then the universe expands, eventually reaches a maximum expansion, turns back, and crashes again.

This may be counterintuitive, because we are in the case where  $k = -1$ , that is the hyperbolic open universe. It is spatially infinite. Yet its evolution is the opposite of what we might have expected. This phenomenon is due to the negative vacuum energy. In other words, the negative cosmological constant causes the universe – instead of expanding exponentially – to expand for a while and come back and collapse. Even though it is spatially infinite, it still comes back and crashes.

So let's be thankful that we don't live in a universe with too large a *negative* cosmological constant. In fact let's also be thankful that we don't live in a universe with too large a *positive* cosmological constant. Either case would be deadly.

In one case the flow would be so large outward that it would take with it and tear apart everything. In the other case we would experience a crunch.

That is the theory of vacuum energy in a nutshell. We investigated several cases, flat and  $\Lambda > 0$ , spherical and  $\Lambda > 0$ ,

and hyperbolic and  $\Lambda < 0$ .

There are more cases. The reader is invited to look into them. Some make sense, some don't. But those which do will have their own characteristic behaviour and be interesting. They can always be analysed by converting them into some sort of mechanical system and then viewing their equation as the conservation of energy for that system.