Part II : Ninth grade

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II.18 Solving a problem with analytic geometry

We are going to study a geometry problem that is difficult to solve using pure geometry but becomes straightforward with analytic geometry. Given an equilateral triangle ABC, and the three lines AD, BE and CF that intersect the opposite sides at the third of their length :

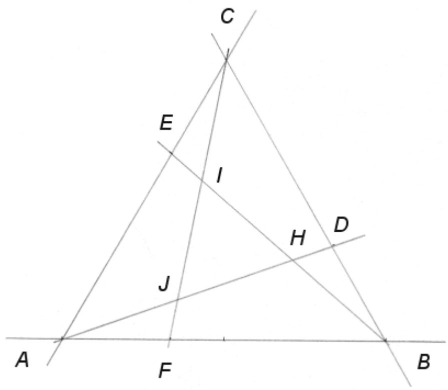

Figure II.18.1 : Equilateral triangle ABC and points D, E, F, at the third of the length of the opposite sides.

The intersections of the three lines AD, BE and CF are named H, I and J.

Theorem : *The triangle HIJ has an area 1/7 of that of ABC.*

We will demonstrate this result for an equilateral triangle ABC by analytic geometry, that is using algebraic coordinates of the various points of the figure in a coordinate system. Then we will generalize to any triangle ABC – because this result is actually true for any triangle ABC – by applying our results on the compression and on the oblique transformation of a triangle (see example 5 p. 355, and exercise II.16.7 p. 357).

Let us imagine a grid in the plane, whose origin is the point A, and the x-axis is the line AB. Without loss of generality, we take the length AB as being the unit.

Coordinates of A, B, C : the coordinates of points A, B and C are :

- $A = (0\,;\;0)$
- $B = (1\,;\;0)$
- $C = (1/2\,;\;\sqrt{3}/2)$

For readers who would have forgotten the height of an equilateral triangle, let h be this height, then by Pythagoras applied to the right triangle obtained by cutting the equilateral triangle vertically in its middle, we have $h^2 + (1/2)^2 = 1^2$.

$$\rightarrow \quad h^2 = 3/4$$

$$\rightarrow \quad h = \sqrt{3}/2$$

<u>Coordinates of D, E, F</u> : Let's now turn to the coordinates of the points D, E and F.

Coordinates of $F = (1/3\,;\ 0)$

Coordinates of D : on the abscissa, by Thales, D has moved back by $1/3$ compared to half of BA, so abscissa $= 5/6$. And on the ordinate, still by Thales, D is one third of that of C, so ordinate $= \sqrt{3}/6$

$D = (5/6\,;\ \sqrt{3}/6)$

Similarly $E = (1/3\,;\ 2/3 \times \sqrt{3}/2) = (1/3\,;\ 1/\sqrt{3})$

<u>Coordinates of H, I, J</u> :

To calculate the coordinates of H, we will use what we learned in the previous lesson on lines in analytic geometry. We will calculate :

- the equations of the lines AD and BE, then
- the coordinates of their intersection

<u>Line AD</u> : It passes through the origin, so its equation is of the form $y = ax$. And it passes through $D = (5/6\,; \sqrt{3}/6)$, so that gives us a constraint on the slope a.

$\sqrt{3}/6 = a \times 5/6$. Hence $a = \sqrt{3}/5$ and the equation of AD is $y = (\sqrt{3}/5) \times x$.

<u>Line BE</u> : Its equation is of the form $y = cx + d$. Since it passes through B, we have $0 = c + d$. And since it passes through E, we have $1/\sqrt{3} = c/3 + d$.

Hence $1/\sqrt{3} = -(2/3)c$

$\rightarrow c = -\sqrt{3}/2$ and $d = \sqrt{3}/2$

Coordinates of point H : It is on both lines

$$\begin{cases} AD: \quad y = \frac{\sqrt{3}}{5}x & (1) \\ BE: \quad y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2} & (2) \end{cases}$$

This is a system of two linear equations with two unknowns. We solve it as we learned to do it : we replace, in (2), y by its value expressed using x, given by (1).

Exercise II.18.1 : Show that the coordinates of H are $\left(\frac{5}{7}\,;\ \frac{\sqrt{3}}{7}\right)$

Now we turn to the line CF and the points I and J.

Line CF : We have the coordinates of points C and F

- $C = (1/2\,;\ \sqrt{3}/2)$
- $F = (1/3\,;\ 0)$

Without going into the same type of calculations as above, we obtain the equation of line CF : $\quad y = (3\sqrt{3})x - \sqrt{3}$.

We can also verify that the two points C and F are on this line and that it is therefore necessarily the right equation. If $x = 1/3$, then $y = 0$, so F is on it. And if $x = 1/2$, then $y = \sqrt{3}/2$, so C is on it.

Coordinates of point J : It is at the intersection of AD and CF. We obtain the system of linear equations with two unknowns

$$\begin{cases} AD: \quad y = \frac{\sqrt{3}}{5}x & (1) \\ CF: \quad y = (3\sqrt{3})x - \sqrt{3} & (2) \end{cases}$$

Exercise II.18.2 : Show that the coordinates of J are $\left(\frac{5}{14}\,;\ \frac{\sqrt{3}}{14}\right)$

Coordinates of the point I : This is the intersection of BE and CF, so its coordinates are the solutions of the system

$$\begin{cases} BE: & y = -\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2} & (1) \\ CF: & y = (3\sqrt{3})x - \sqrt{3} & (2) \end{cases}$$

Exercise II.18.3 : Show that the coordinates of I are $\left(\frac{3}{7} \,;\, \frac{2\sqrt{3}}{7}\right)$

Summary of the coordinates of H, I and J :

- $H = \left(\frac{5}{7} \,;\, \frac{\sqrt{3}}{7}\right)$
- $I = \left(\frac{3}{7} \,;\, \frac{2\sqrt{3}}{7}\right)$
- $J = \left(\frac{5}{14} \,;\, \frac{\sqrt{3}}{14}\right)$

Calculation of the distance HI : Generally speaking, using analytic geometry, the distance between any two points H and I, with coordinates $(x_1 \,;\, y_1)$ and $(x_2 \,;\, y_2)$, is calculated using the Pythagorean theorem.

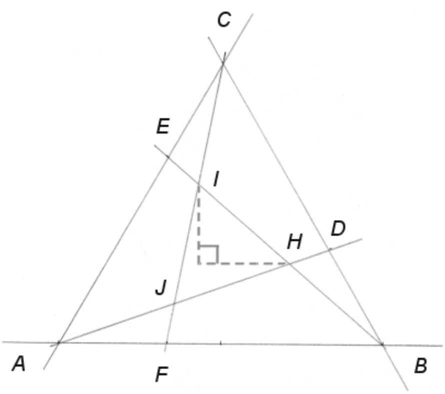

Figure II.18.2 : Calculation of the distance HI by the Pythagorean theorem.

We obtain

$$(\text{distance } HI)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Exercise II.18.4 : Show that $HI = 1/\sqrt{7}$.

As the whole figure is invariant under rotation of 120° around the center, this is also true of the distances IJ, and JH.

Therefore the small triangle HIJ is equilateral, and it is obtained by multiplying all the dimensions of the triangle ABC by $1/\sqrt{7}$.

So its surface area is in the square of this ratio, that is to say

$$\text{surface area of } HIJ = 1/7 \times \text{surface area of } ABC$$

Q.E.D.

Arbitrary triangle : To conclude, we have seen that a compression or an oblique transformation do not change the ratios of areas. Given that any triangle can be obtained from an equilateral triangle by compression and oblique transformation, the theorem remains true for any triangle.

Exercise II.18.5 : On a sheet of graph paper, draw an arbitrary triangle ABC. Plot the points D, E, F, then H, I, J.

Carefully identify the coordinates of H, I and J.

Use any software for calculating the area of a triangle, such as those found on the Internet, to verify that the area of HIJ area of ABC divided by 7.

Exercise II.18.6 : 12th grade level : prove the theorem using barycentric coordinates.

Hint : See our book *High school mathematics : vol 3 12th grade*.

Exercise II.18.7 : Consider triangle ABC whose vertices have coordinates $A = (0,\ 0)$, $B = (24,\ 0)$, $C = (18,\ 12)$.

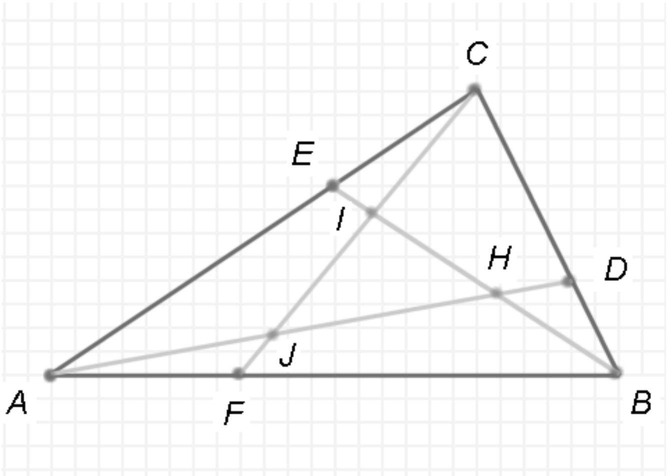

Figure II.18.3 : Triangle ABC and points D, E, F at one third of the length of each side.

Is triangle ABC right-angled at C ?

Hint : In a right triangle at C, the height drawn from C and intersecting AB at point K (not shown in the figure) is such that $CK^2 = AK \times KB$.

What is the area of ABC ?

What are the coordinates of D, E, and F ?

The points H, I, J are the intersections of the segments shown in the figure. What are the coordinates of H, I, and J ?

Use an online triangle calculator to verify that the area ofHIJ is one-seventh of the area of ABC.

Exercise II.18.8 : Let ABC be an arbitrary triangle and G its center of gravity.

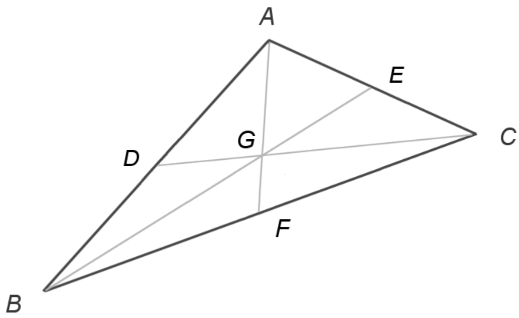

Figure II.18.4 : Arbitrary triangle ABC and its center of gravity G, that is, the intersection of the medians.

Show that the three triangles BGC, BGA, and AGC have the same surface area.

Show that the six triangles ADG, BDG, BGF, FGC, CGE, and EGA have the same surface area.

II.19 Parameterized collection of lines

II.19.1 Introduction

The aim of this "lesson-exercise" is to continue to develop our understanding and mastery of lines, that is, linear relations. We will also learn the concept of parameter, and of parameterized family of objects.

A line equation, in its general form, is

$$ax + by + c = 0 \tag{II.19.1}$$

The line "defined by this equation" is the set of points $(x,\ y)$ in the plane, such that $ax + by + c = 0$.

For example, the locus of the points $(x,\ y)$ satisfying the equation $2x - 3y + 5 = 0$ is drawn below. It is a line. To plot it, we recall that we just need to find two easy points (for example $x = 5 \ \rightarrow \ y = 5$, and $y = 0 \ \rightarrow \ x = -5/2$), and we have our line

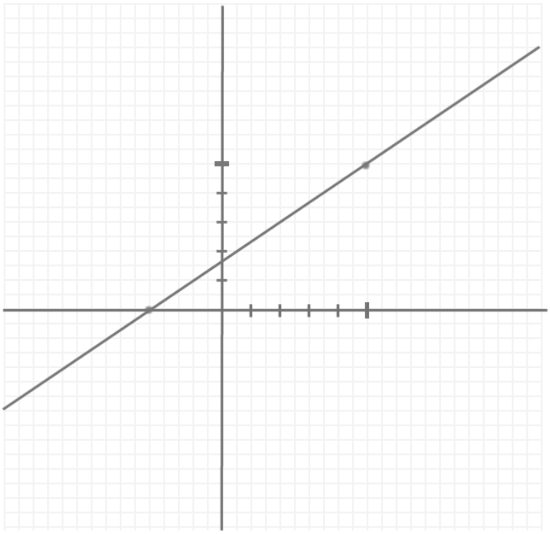

Figure II.19.1 : Line with equation $2x - 3y + 5 = 0$.

II.19.2 Parameterization

Let us now suppose that the coefficients a, b and c are of the form

- $a = 2 + 2t$
- $b = -3 + t$
- $c = 5 - t$

where t is an arbitrary *parameter* (i.e. a number).

Then the coefficients a, b and c are functions of t. By varying t, we obtain a whole collection of straight lines. The straight line in figure II.19.1 corresponds to $t = 0$.

We speak of a parametric, or parameterized, family $\mathcal{D}$ of lines, where each line D_t has the equation

$$a(t)\,x + b(t)\,y + c(t) = 0 \qquad \text{(II.19.2)}$$

with

- $a(t) = 2 + 2t$
- $b(t) = -3 + t$
- $c(t) = 5 - t$

Let us draw another line from the family $\mathcal{D}$. Let $t = 1$. The equation of D_1 is $4x - 2y + 4 = 0$. The following points are on it :

$x = 0 \;\to\; y = 2$

$x = 2 \;\to\; y = 6$

$x = -2 \;\to\; y = -2$

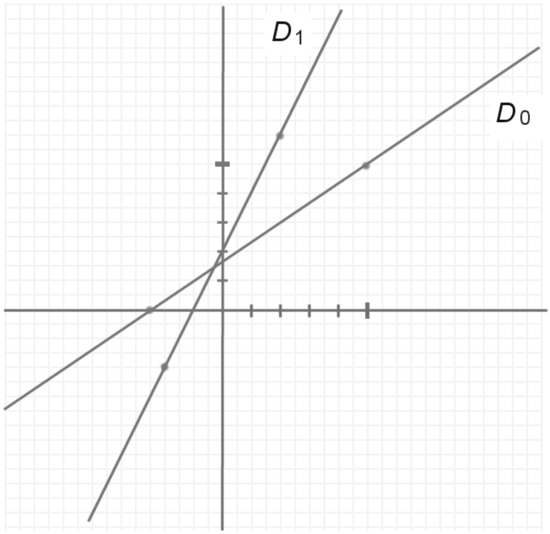

Figure II.19.2 : Lines D_0 and D_1.

II.19.3 Point of intersection of D_0 and D_1

Let us call I the point of intersection of D_0 and D_1. We know how to find its coordinates (x, y) : we solve the system of equations

$$\begin{cases} 2x - 3y + 5 = 0 & \quad (t = 0) \\ 4x - 2y + 4 = 0 & \quad (t = 1) \end{cases}$$

Solution : let's multiply the 1st equation by -2

$$\rightarrow \ -4x + 6y - 10 = 0$$

And let's add the 2nd equation to eliminate the term in x, and obtain y

$$\rightarrow \ 4y - 6 = 0$$

Whence $y = 3/2$. Then we find $x = -1/4$. That is, the intersection point I has coordinates $(-1/4\,;\ 3/2)$.

II.19.4 Study of a third line of the parametric family

Let's look at a third line of the family $\mathcal{D}$. Let's take $t = 2$. The equation of D_2 is $6x - y + 3 = 0$. We find some points on it, in order to draw it :

$x = 0 \ \rightarrow \ y = 3$

$y = 0 \ \rightarrow \ x = -1/2$

$x = 1/2 \ \rightarrow \ y = 6$

$x = -1 \ \rightarrow \ y = -3$

Hey, hey, on figure II.19.3 the point of intersection with D_0 seems to be the same !

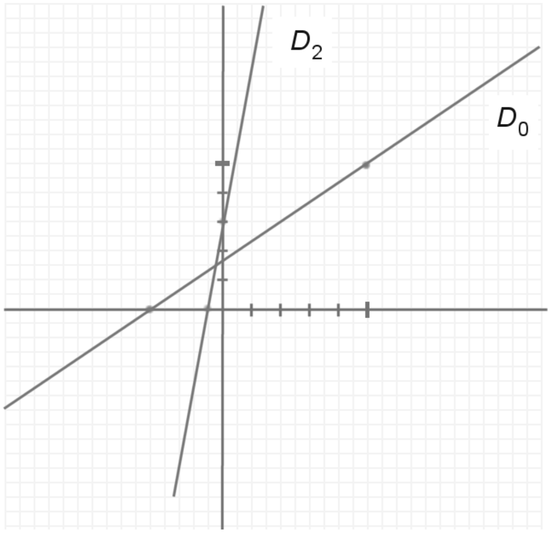

Figure II.19.3 : Lines D_0 and D_2.

This can also be verified with calculation, by solving

$$\begin{cases} 2x - 3y + 5 = 0 & (t = 0) \\ 6x - y + 3 = 0 & (t = 2) \end{cases}$$

> **Exercise II.19.1** : Solve this system.

II.19.5 Why the intersection point is the same

Heuristic explanation : $\mathcal{D}$ is a very simple family of lines. A very simple family of lines can only consist of lines that all pass through the same point or are all parallel. It can't be a Pick-up Sticks !

Calculus : All lines in $\mathcal{D}$ have the equation

$$(2 + 2t)x + (-3 + t)y + (5 - t) = 0 \qquad \text{(II.19.3)}$$

This equation can be rewritten by separating terms independent of t from those with t as a factor :

$$2x - 3y + 5 + t(2x + y - 1) = 0 \tag{II.19.4}$$

Therefore all lines, whatever t, necessarily pass through the solution of

$$\begin{cases} 2x - 3y + 5 = 0 \\ 2x + y - 1 = 0 \end{cases} \tag{II.19.5}$$

That is, $x = -0.25$ and $y = 1.5$.

Exercise II.19.2 : Verify that $x = -0.25$ and $y = 1.5$ is the solution of the system

$$\begin{cases} 2x - 3y + 5 = 0 \\ 2x + y - 1 = 0 \end{cases}$$

Note that in $\mathcal{D}$ there is one missing line. It is precisely the line $2x + y - 1 = 0$, i.e. the one with t as a factor in equation (II.19.4).

It is missing in the same way as in equations of the form $y = ax + b$, the vertical lines are missing.

The line $2x+y-1 = 0$ would correspond in $\mathcal{D}$ to $t = +\infty$. Indeed when t is very large, the part that is not a factor of t becomes negligible. We can divide the whole equation (II.19.4) by t to convince ourselves of this. But $+\infty$ is not a number, it is just a way of saying : this line is a “limit” line, and is not in $\mathcal{D}$.

Exercise II.19.3 : Redo the entire lesson with a different set of parameters $a(t)$, $b(t)$, and $c(t)$ of your choice, each defined as a linear function of t

II.19.6 Power of the concept of parameterization

We have just seen the technique of parameterization applied to the coefficients of a line to create, with linear functions[1] of a parameter t, a collection of lines.

This technique proves to be very powerful.

On the one hand, it allows us to create collections of lines that are more complex than those where all lines pass through the same point. For instance, by taking

$$a(t) = 2t,\ \ b(t) = -1,\ \text{ and } c(t) = -t^2$$

we obtain a family of lines that are not concurrent but have an interesting structure.

> **Exercise II.19.4** : Plot the lines with coefficients $a(t) = 2t$, $b(t) = -1$ and $c(t) = -t^2$ for some values of t.
>
> What can we say about the geometry of this family of lines ?

On the other hand, we have seen that the functions $y = f(x)$ cannot correspond to all the curves in the plane, because each x must correspond to only one y. Thus a circle is not a curve in the plane with an equation of the form $y = f(x)$.

Of course, we can decompose it into two semicircles symmetrical with respect to the x axis, and each having such a formula. But the technique of parameterization is much simpler to handle loci that are not functions $y = f(x)$.

All points with coordinates

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \tag{II.19.6}$$

where t is a parameter varying between 0 and 2π form a circle centered at the origin and of radius 1.

1. We use the adjective linear for all functions of the form $ax + b$, not only ax, even though *stricto sensu* linear only applies to the second relation, the first being affine.

We can similarly define curves that are more complicated than a circle using a parameterization of the coordinates of their points.

For example, if a and b are real numbers and θ is the parameter,

$$\begin{cases} x = \dfrac{a}{2} + b\cos\theta + \dfrac{a}{2}\cos 2\theta \\ y = b\sin\theta + \dfrac{a}{2}\sin 2\theta \end{cases} \tag{II.19.7}$$

is the set of equations of a *Pascal's limaçon*, discovered by Étienne Pascal (1588, 1651), the father of Blaise Pascal (1623, 1662).

Taking $a = 2$ and $b = 1$, here are sixty-three points on the curve calculated by varying θ from 0.1 to 6.3 in increments of size 0.1 radians.

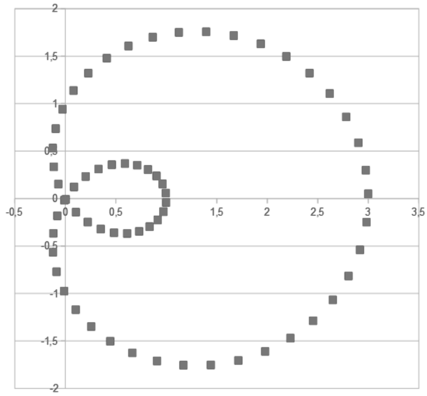

Figure II.19.4 : Pascal's limaçon.

Exercise II.19.5 : Draw Pascal's limaçon for several different pairs of parameters a and b. Try $(a,\ b) = (1,\ 1)$, $(a,\ b) = (2,\ 1)$, $(a,\ b) = (3,\ 1)$, etc.

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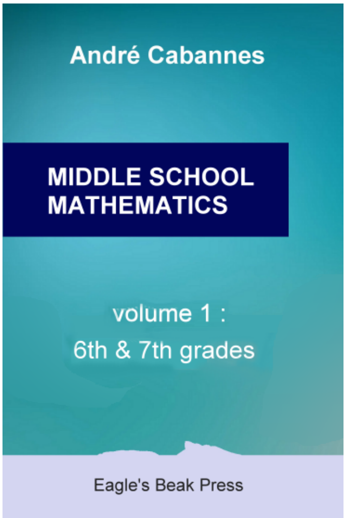

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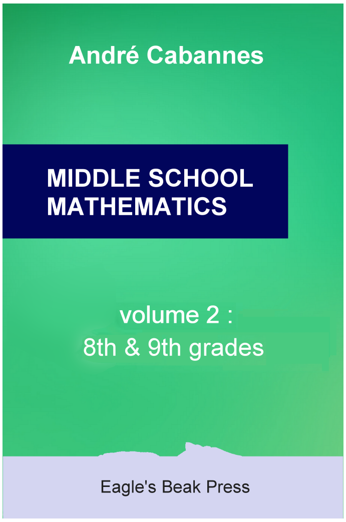

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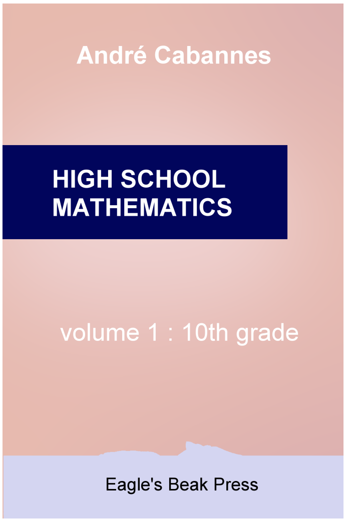

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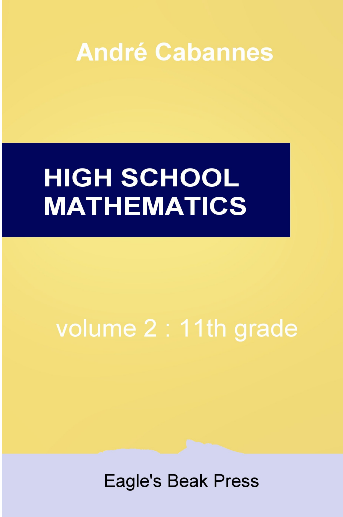

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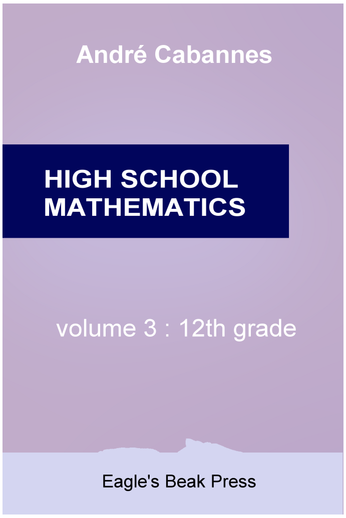

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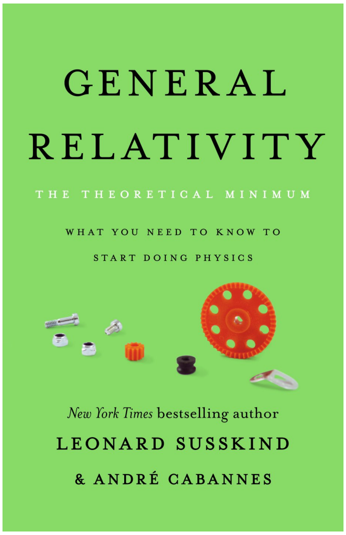

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