Part I : Sixth grade

Contents

I.34 Splitting a cube into three identical pyramids

I.34.1 Introduction

We will construct a pyramid that forms part of a cube, using cardboard, scissors, and glue. The pyramid will have the bottom face of the cube as its base and one of the top corners as its vertex.

We will see that the volume of this pyramid is exactly one-third of the cube's volume. Better than that : we will be able to reassemble the cube using three identical pyramids.

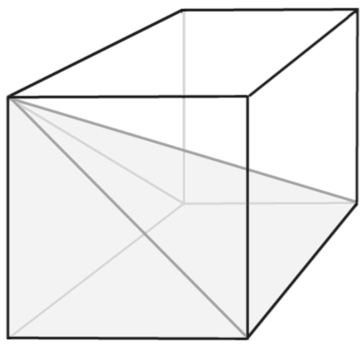


Figure I.34.1 : Square-based pyramid inscribed in a cube.

This small "mathematical experiment" has important implications. It provides insight into the nature of three-dimensional space and the calculation of volumes.

We will see that it leads to a three-dimensional generalization of the following familiar facts in the plane :

1. Two identical right-angled isosceles triangles can be used to form a square.
2. The area of any triangle is the product of the length of a side and the perpendicular height, divided by two.

I.34.2 Pyramid pattern

To build the pyramid, we start with this pattern[1] :

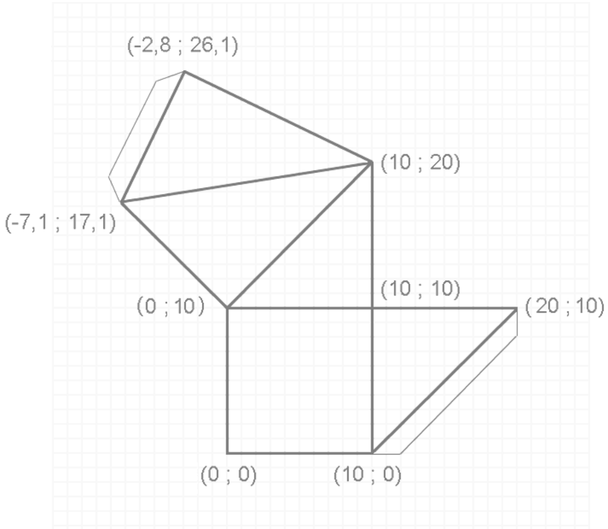


Figure I.34.2 : Pattern to build the pyramid.

I.34.3 Construction of three identical pyramids and assembly of the cube

We cut according to the pattern and start folding.

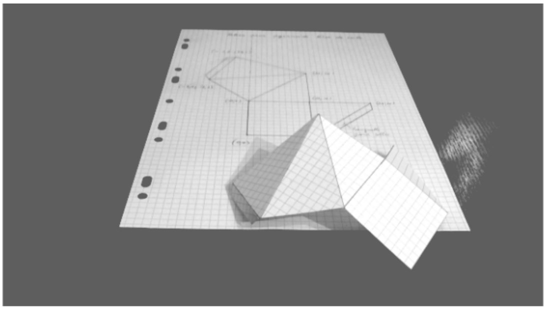


Figure I.34.3 : Construction step (1).

1. Use cardboard sheets that are ruled, i.e. they have a printed grid.

We complete the first pyramid.

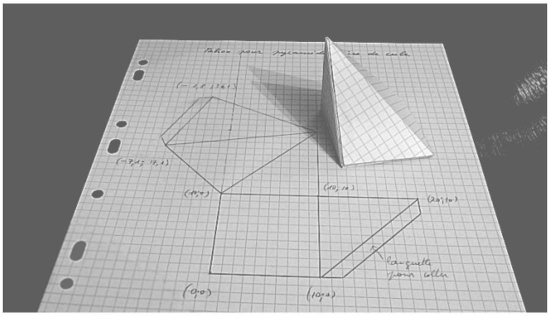


Figure I.34.4 : Construction step (2).

We make three pyramids with the same pattern (also called a "template" or a "model"). And we begin to adjust them to make a cube.

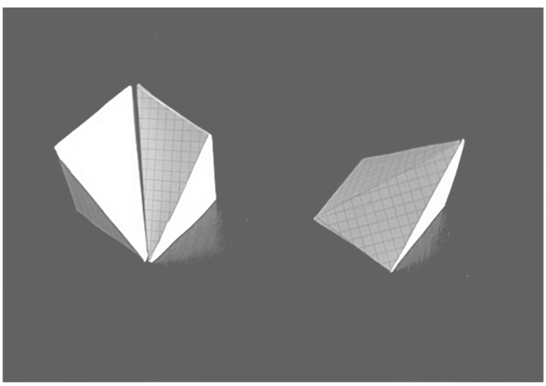


Figure I.34.5 : Construction step (3).

The three pyramids fit together like three pieces of a three-dimensional puzzle to make a cube.

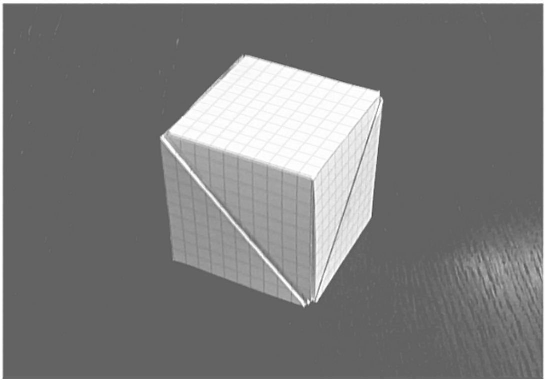


Figure I.34.6 : Construction step (4).

A question arises : is there any void inside the cube, or do the three pyramids fill it completely ? It is easy to see that, since the slanted faces of the model pyramid (fig. I.34.1) form a 45° angle with the base, it is impossible for any void to exist inside.

Thus, the cube is indeed filled precisely and completely by the three pyramids. This leads us to important volume calculations.

I.34.4 Volume of the pyramid

Since the three identical pyramids fill the cube, the initial pyramid has a volume equal to one third of that of the cube. It is therefore the surface area of its base multiplied by its height divided by 3.

We can write :

$$V = S \times h \times \frac{1}{3} \tag{I.34.1}$$

where V is the volume of the pyramid, S the area of its base, that is to say one of the faces of the cube, and h the height, that is to say one edge of the cube.

We will use this result to extend it to any pyramid, and even to any cone.

This folding lesson plays an important role in all of our courses from 6th to 12th grade. In the same way that any triangle has as its area its base multiplied by its height divided by two, any pyramid or cone has as its volume its base (i.e., the area of its base) multiplied by its height divided by three.

I.34.5 Volume of an arbitrary pyramid and an arbitrary cone

To generalize formula (I.34.1), first, to an arbitrary pyramid, we will use our familiar technique of cutting the object we are working with into thin horizontal slices, as we have already done with a triangle (fig. I.26.8, page 194).

Now we know the volume of the pyramid in figure I.34.4. It has a face of the square as its base, and one of its edges is vertical and is also one of the edges of the cube. We can cut this pyramid into thin horizontal slices like pizza boxes.

Then we can move the pizza boxes so as to make a pyramid whose apex is in the middle of the upper face of the cube. Just as in figure I.26.8, p. 194, when we slide the horizontal slices we can keep a triangle, here it is easy to show that we can keep a pyramid.

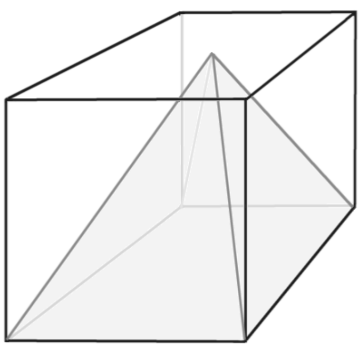


Figure I.34.7 : New pyramid obtained from the one in figure I.34.1 by sliding the horizontal slices so that they are all centered above the center of the bottom face of the cube.

This does not change the volume of the pyramid. It remains equal to its base multiplied by its height divided by 3. We can even slide the top of the pyramid anywhere on the plane of the upper face of the cube without changing the volume of the pyramid.

Now let's see what happens when, starting from the initial pyramid, we stretch it vertically, keeping its vertex aligned with the vertical edge of the cube. For example, let's double the height of the pyramid (keeping the same base).

Its new volume will be double that of the original pyramid. Why ? One way to see this is as follows : imagine the original pyramid as composed of a very large number of small cubes. To create the new pyramid with double the height, we simply replace each small cube with one that is double its height. Thus, the new pyramid will have double the volume. Therefore, formula (I.34.1) still applies :

$$V = S \times h \times \frac{1}{3}$$

Exercise I.34.1 : Show that the pyramid of height 3 units, where 1 unit is the length of an edge of the bottom cube, and having as base the lower face of this cube, has the same volume as the bottom cube.

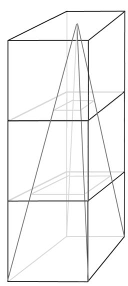


Figure I.34.8 : Pyramid three times higher than the bottom cube. Its volume is the same as the volume of this cube.

Exercise I.34.2 : By dividing the volume of the pyramid of the previous exercise into 12 identical small square-based pyramids plus two rectangular parallelepipeds with square bases, could you calculate the volumes of the portions of the pyramid contained respectively in the first cube, in the second cube, and in the third cube ?

Exercise I.34.3 : Start with a unit cube as in figure I.34.1. Cut it into 6 pyramids, each having a face of the cube as its base and the center of the cube as its vertex. Deduce another proof of formula (I.34.1).

Just as we can slide or stretch the slices of the initial pyramid to make a pyramid with a square base inclined as we want and of any height, we can also make a pyramid with *any base* using an assembly of a *very large number* of thin pyramids with a square base. Therefore the volume formula also applies to any pyramid.

Finally, we can remodel each horizontal slice to make a disk. We then produce a stack of disks with regularly decreasing radii forming a cone. Consequently, the volume formula still applies to cones. [2]

I.34.6 Geometry and numbers

We can do geometry without using numbers. But we can also discover connections between geometry and arithmetic, and it's a lot of fun.

Here's an example. Consider a cube with an edge length of 1 meter. Each edge can be divided into a thousand millimeters. The cube itself can be thought of as a stack of a

2. The essence of these manipulations is that, once we have shown that a thin square-based pyramid of any height and any obliqueness satisfies the formula $V = S \times h/3$, we can use many such pyramids to form pyramids with any base, and even create cones with all sort of bases and oblique shapes. The formula for the volume still holds.

thousand layers, each with an area of 1 m^2 and a height of one millimeter. Each layer can be seen as consisting of a thousand by a thousand small cubes with 1-millimeter sides. The entire cube is therefore an assembly of a thousand by a thousand by a thousand small cubes measuring 1 mm by 1 mm by 1 mm, that is : a billion small cubes.

Now, with these 1-millimeter cubes, we will build a pyramid like the one at the beginning of the lesson. It won't be perfect, as some faces will not be entirely smooth, but it will be close – much like the pyramids of Egypt.

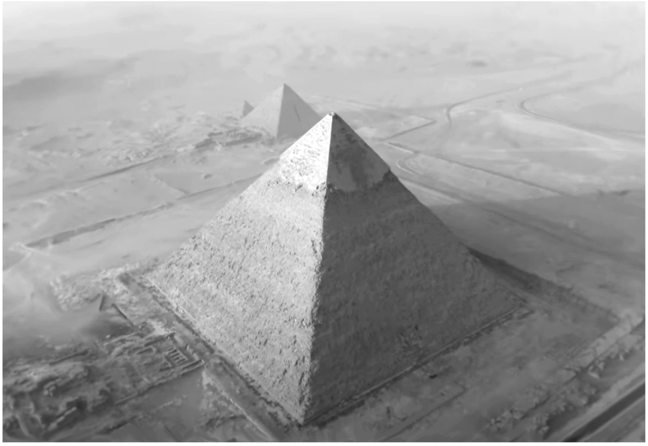


Figure I.34.9 : Great Pyramid of Cheops near Cairo in Egypt. Source : National Geographic.

Here's how to do it : first, on a base of one meter by one meter, we arrange a million small cubes to create a first layer that is one meter square and one millimeter high. Next, we place a second square layer consisting of only 999 by 999 small cubes, aligned along one of the vertical edges. [3] Then we add a third layer of 998 by 998 small cubes, and so on, until we reach a single small cube on the thousandth layer.

3. It will not have exactly the same shape as the Great Pyramid of Cheops, as it will have a vertical edge. However, as we saw earlier, this does not affect the volume. The volume is the same for the pyramid in figure I.34.1 and the pyramid in figure I.34.7.

This forms practically the same pyramid as the one built in the lesson. So its volume is almost exactly 1/3 of a cubic meter. In other words, it contains almost exactly 1 000 000 000/3 small cubes. So we have shown that the sum of the squared numbers from 1 to 1000 is almost equal to a third of a billion, which we write

$$1 + 4 + 9 + 16 + ... + 999^2 + 1000^2 \approx \frac{1\,000\,000\,000}{3}$$

In fact, we can check with a spreadsheet that

$$1 + 4 + 9 + 16 + ... + 999^2 + 1000^2 = 333\,833\,500$$

The exact formula for the sum of the squares from 1 to n is

$$1 + 4 + 9 + ... + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(I.34.2)}$$

We can verify that, for $n = 1000$, this gives the number 333 833 500.

The ancient Greeks did geometry using numbers as little as possible to measure distances, because the Pythagorean School in the 6th century BC discovered that some lengths could not be expressed as fractions of each other. Specifically, if we consider a square with a side of 1 meter, its diagonal has a length that cannot be written as a fraction a/b, where a and b are whole numbers. This length is approximately 1.414 meters, but not exactly. Later, we will demonstrate that no fraction can represent it precisely.[4]

Similarly, if our cube has an edge of one meter, its "major diagonal" in space is not a fraction of 1. It is approximately 1.732 m, but not exactly.

4. We will study this in 9th grade, but the idea is quite simple : if a/b were the length of the diagonal, we could show, using the Pythagorean theorem, that $a^2/b^2 = 2$, so $a^2 = 2b^2$. This leads to a contradiction, as there is not the same number of factors of 2 on both sides. Therefore, the diagonal cannot have the length a/b, where a and b are whole numbers.

This discovery convinced the Greeks to avoid as much as possible the concept of length *expressed numerically* in geometric reasoning, even if of course we continued to *compare* lengths, and to measure distances (for example that of Marathon to Athens which is about 40 kilometers), fields, volumes, etc. This is the reason why Greek geometry is a "pure geometry", i.e. which makes little use of numerical lengths and doesn't use at all coordinates in a Cartesian reference frame.

The discovery that some distances could not be expressed as a fraction of the length chosen as the unit of measurements – what is called the "incommensurability" of certain lengths – also upset Pythagoras, who believed – in a rather vague way – that "everything was number", including in music.

Pythagoras was a genius bordering madness. He ordered the discovery to be concealed. The secrecy was so effective that Democritus, born more than 30 years after Pythagoras's death, could still believe that "line segments were composed of a finite number of atoms" – a notion that leads to contradictions with the "incommensurability" of certain lengths.

But science advanced. Euclid wrote a treatise that remained influential for over 2000 years. Apollonius studied conics. The Late Alexandrian School introduced, among other things, projective geometry, the theorems of Pappus and Menelaus – these are only the most notable milestones. The Arabs developed algebra and trigonometry. The Middle Ages and the Renaissance in Europe also made significant contributions.

As for the "infinitely small" (such as the "very small cubes" with which we have just constructed a pyramid), further progress came only in the 17th century in Europe, through the work of Leibniz and Newton, who built upon the important contributions of many scholars before them, to start with, Descartes.

It is a fascinating story, and the construction of a body of knowledge that is even more so : it explains much of how the world around us works – at least the world outside of living things. Indeed, the living world largely remains beyond the reach of purely mathematical and physical explanations.

The study of life and living organisms is a discipline in itself, called *biology*. The science of treating human illness is called *medicine*.

We admire mathematics, but we are not blind to its limitations. There are other fields where mathematics contributes little to explaining things : we've just mentioned biology and medicine ; there is also sociology (the study of human behavior in society [5]), politics, history, law, and the literature of great civilizations (Anglo-Saxon, Spanish, Chinese, French, Russian, etc.).

For the curious mind – like that of any child not stifled by a passive, conformist education focused on rote learning – the world is an Ali Baba cave of wonders that one lifetime is not enough to explore.

Exercise I.34.4 : For a few values of the number n, verify the formulas

$$1 + 4 + 9 + ... + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$1 + 3 + 5 + 7 + ... + (2n-3) + (2n-1) = n^2$$

Exercise I.34.5 : Knowing that the height of an equilateral triangle is approximately 0.866 times the length of a side, what is approximately the length of a side of an equilateral triangle of area 1 m^2 ?

Hint : Try 1 m. See that it is too short. Try 2 m. See that it is too long. Try 1.5 m. Is it too short or too long ? Continue a few steps, each time taking the average of the number that is too short and the number that is too long.

5. A book we are a great fan of is Peter L. Berger's, *Invitation to Sociology : A Humanistic Perspective*, Anchor, 1963.

Exercise I.34.6 : Knowing that the height of a regular tetrahedron is approximately 0.8165 times the length of an edge, what is approximately the length of an edge of a regular tetrahedron of volume 1 m^3 ?

Hint : Follow the same method by successive approximations as in the previous exercise.

Exercise I.34.7 : Show that the center of a regular tetrahedron is at a quarter of the height joining a face to a vertex, starting from the face.

Hint : Draw inspiration from exercise I.34.3.

Exercise I.34.8 : Consider four points in space with the following coordinates (see fig. I.31.7, p. 231) :

$$A = (0,\ 0,\ 0)$$
$$B = (3,\ 0,\ 0)$$
$$C = (1,\ 3,\ 0)$$
$$D = (2,\ 2,\ 5)$$

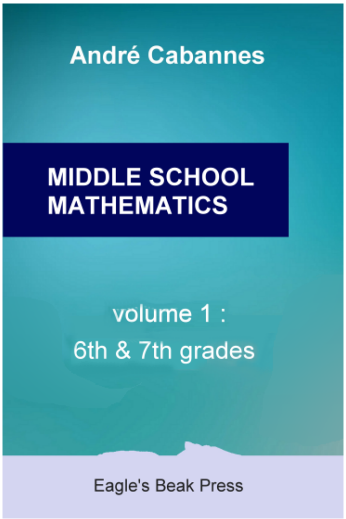
What is the volume of the pyramid they form ?

Hint : Show that the volume is not changed by replacing point D with $D' = (0,\ 0,\ 5)$. Also replace C with another more convenient point.

Exercise I.34.9 : Explain schematically how one could calculate the volume of a regular dodecahedron (see fig. I.33.9 and I.33.10) by dividing it into pyramids with a triangular base.

What intermediate data do we need to calculate to obtain the volume of a regular dodecahedron with an edge of 1 meter ?

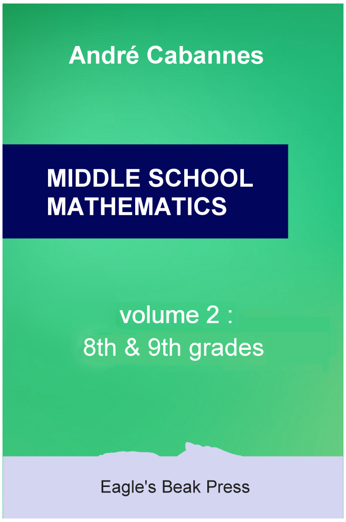
English titles by André Cabannes



www.amazon.com/dp/2958738558
Middle school mathematics

Volume 1 : 6th & 7th grades

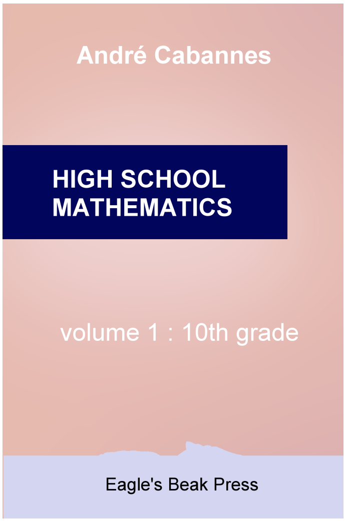
for middle school students and their parents



www.amazon.com/dp/295873854X
Middle school mathematics

Volume 2 : 8th & 9th grades

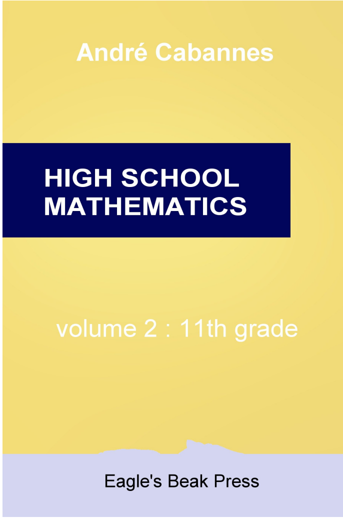
for middle school students and their parents



www.amazon.com/dp/2958738531

High school mathematics

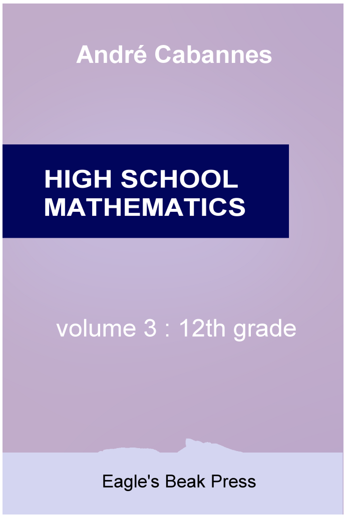
Volume 1 : 10th grade



www.amazon.com/dp/2958738523

High school mathematics

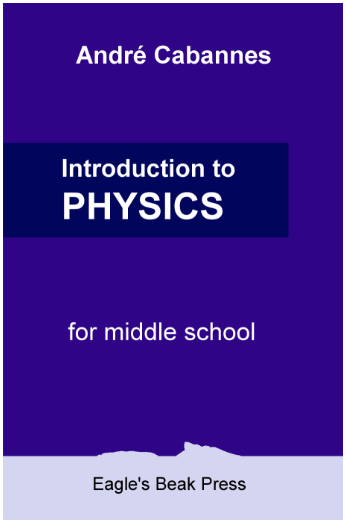
Volume 2 : 11th grade



www.amazon.com/dp/2958738515

High school mathematics

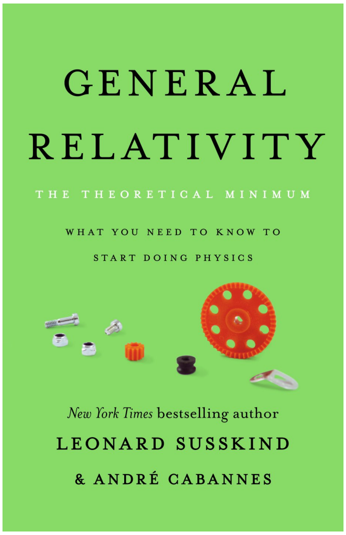
Volume 3 : 12th grade



www.amazon.fr/dp/B0G5JTM7RP

Introduction to physics

Middle school



www.amazon.com/dp/B09ZB613QY

General Relativity

Graduate studies.