Lesson 10 : Gravity waves

Notes from Prof. Susskind video lectures publicly available on YouTube

Introduction

The questions we want to address in this lesson are weak gravitational fields, linearity versus non-linearity, gravitational waves (aka gravity waves). We will finish with an overview of how Einstein field equations can be derived from the action principle.

Again working out the equations of general relativity is always unpleasant. We are not going to do it here. The calculations would fill pages even for simple things. And they probably would not be terribly illuminating. To learn the subject you really have to compute yourself and solve the equations and so forth. You just have to sit down and do it on your own.

On the other hand the principles are straightforward. And it is easy enough to explain what you get when you do solve equations. So that is the way we will talk about gravity waves : by writing down the equations and then writing their solutions.

To start with, we are interested in what could be called weak gravitational waves.

Weak gravitational fields

We talk of weak gravitational field when the gravitational variations over space-time are small enough that we can make approximations, such as the variations *squared* of the gravitational field can be taken to be zero.

When a quantity is small and we expand an equation about the smallness of that quantity, the usual rule is to ignore things which are of higher order than one in that small quantity. The typical example is using the beginning binomial theorem

$$(1+\epsilon)^n \approx 1 + n\epsilon$$

Okay so we start from an equilibrium. And we are going to talk about fluctuations, or perturbations, about the equilibrium situation. We will look at the simplest equilibrium solution of Einstein equations.

Let's write Einstein equations first. And let's take the case without any matter – no energy-momentum tensor. Then the equations of motion are just

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, \mathcal{R} = 0 \tag{1}$$

It can be simplified. If we take the trace of both sides, after having raised one index upstairs, since $\mathcal{R}$ is the trace of the Ricci tensor $\mathcal{R}^{\mu}_{\nu}$, and 4 is the trace of g^{μ}_{ν} (which is the Kronecker delta), this gives $\mathcal{R} = 0$.

Therefore we can ignore the second term in equation (1). And it becomes simply

$$\mathcal{R}_{\mu\nu} = 0 \tag{2}$$

It doesn't really matter because we are not going to write down the details anyway. But equation (2) is Einstein's equation in a context where there is no energy-momentum tensor on the right hand side of the full equation (see equation (36) of chapter 9).

What is an equilibrium situation ? It is, first of all, a solution which has no time dependence. And the solution also doesn't have any matter on the right hand side – matter being another word for the energy-momentum tensor.

There is really only one equilibrium situation. It is just empty space is time independent. By empty space we mean empty flat space, no curvature, no interesting gravitational field. And in that case, the metric $g_{\mu\nu}$ has a simple form.

Remember that it is improper to say "the metric is equal to such and such matrix", because the metric is a tensor. In each coordinate system, it has a different expression with components[1]. If I wrote that the metric of a flat plane, in Euclidean geometry, is just the Kronecker delta symbol, you would correct me : "no, the metric of a flat plane is not the Kronecker delta." What is true is that a possible expression of the flat metric, with an appropriate coordinate system, is the Kronecker delta. But if I used other coordinates, the expression of the metric would not be the Kronecker delta. We could use curved coordinates, use polar coordinates, use any other kind of coordinates.

What is special about flat space, in Euclidean geometry, or

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1. It is the same for vectors in 3D Euclidean space. A vector *is not* a set of three numbers. It is a direction in space, which, *in a given basis*, has an expression with three numbers. And in another basis, the collection of three numbers will be different. The same is true of tensors. They are abstract creatures. And they have expressions with multi-indexed collections of components, which depend on the coordinate system chosen to describe space-time and the objects in it.

in Riemaniann geometry, is that you can find coordinates in which the metric has a nice simple form.

The same is true in general relativity – in which, as we know, the geometry is Minkowski-Einstein geometry [2]. For flat space-time with no gravitational field, there is a choice of coordinates in which the metric $g_{\mu\nu}$ has a simple form

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

The first row and the first column correspond to time. So the four columns correspond to $(t,\ x,\ y,\ z)$. Same for the four rows.

This is the metric of flat space-time in the most appropriate coordinate system.

And it is an equilibrium solution. That is, it is a solution of the Einstein field equations, and this solution doesn't depend on time.

Remember that the complete description of motion is : "the field tells the sources how to move" (they move along geodesics) and "the sources tell the field what to be" (it is a solution to Einstein field equations, which generalize Poisson's equation of the Newtonian case, see equation (4) of chapter 9). In general the solution is time dependent of course. But

2. At any point in space-time the signature of the metric at that point is three plus signs for the three spatial coordinates and a minus sign for the time coordinate.

here we are interested in an equilibrium solution – therefore independent of time. Secondly we are interested in the equilibrium solution for the simplest type of source configuration, namely, no sources.

So we are looking at the case where Einstein field equations reduce to $\mathcal{R}_{\mu\nu} = 0$. They say that certain components of curvature are equal to 0. It is a purely geometric condition.

And we have a solution which is particularly simple : no curvature at all anywhere. It is easy the check that it satisfies $\mathcal{R}_{\mu\nu} = 0$. This concludes the first step toward weak gravitational waves.

Now let's think about a space-time which is close to empty space-time.

Suppose that, very far away from where we are, some complicated phenomenon is happening. For example a binary pulsar is rotating rapidly. And it emits some complicated gravitational waves. Near the pulsar the gravitational field may be very strong. Even the gravitational waves [3] might be rather strong.

But if you go far enough away the gravitational radiation, that is, the gravitational waves that are produced by this thing, are going to be very weak.

What does weak mean ? Weak means that the metric can be chosen – again, I emphasize, in the appropriate coordinate system – to be equal to $\eta_{\mu\nu}$ plus something small. Small

3. That is the evolution of the field over space-time, like the moving ripples created by a stone thrown in a pond.

means that the components of the small thing are much smaller than those of η. So we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

As far as I know, it is called h because it is the letter that comes after g.

Unlike $\eta_{\mu\nu}$, the perturbation term $h_{\mu\nu}$ is in general a function of position. When we say "position" we mean the four coordinates of an event in space-time of course. $h_{\mu\nu}$ varies from place to place and also from time to time, and it might describe a wave.

We will come back to waves in a moment. But let's first ask ourselves : what are we going to do with this $h_{\mu\nu}$?

Answer : we will take the metric given by equation (4), calculate its Ricci tensor $\mathcal{R}_{\mu\nu}$, and set it equal to zero. That will give us an equation for h.

The equation that we actually get for h is not big enough to fill a whole page, but it is sufficiently big to be unpleasant. So I'm just going to be schematic. I'm going to show you what goes into it.

First of all, let's look a the Ricci tensor $\mathcal{R}_{\mu\nu}$. It is really a combination of components of the Riemann curvature tensor $\mathcal{R}_{\mu\nu\tau}^{\ \ \ \sigma}$. I'm not going to write the full curvature tensor, I'm just going to remind you what it contains, that is, its structure. So the $\mathcal{R}$ below means the full rank 4 curvature tensor. It has the following structure

$$\mathcal{R} = \partial\Gamma + \Gamma\Gamma \tag{5}$$

It contains first derivatives of the Christoffel symbols, and it contains the Christoffel symbols quadratically, that is products of two Christoffel symbols. I could put the indices everywhere, spell out the various terms which look like $\partial\Gamma$, and the various terms which look like $\Gamma\Gamma$. But expression (5) giving the structure is really all we need.

What about the Christoffel symbol Γ ? Likewise its structure is

$$\Gamma = \frac{1}{2} g^{-1} \partial g \tag{5}$$

The symbol g^{-1} means the inverse matrix of g, or equivalently the metric tensor with two contravariant indices.

Just like g with covariant indices can be expanded as in equation (4), g^{-1} can also be expanded in powers of h. The first contribution to it is just the inverse of the η symbol. But that is η itself, because η is its own inverse. And then we simply have the same h term but with a minus sign (just like $(1/(1+\epsilon) \approx (1-\epsilon))$. So the expansion of g^{-1} up to the first perturbation term is, in matrix notation,

$$g^{-1} = \eta - h \tag{6}$$

where actually the same matrix h appears with a minus sign.

Now, dropping the 1/2 because we are only looking at structures, equation (5) becomes

$$\Gamma \sim (\eta - h)\partial h \tag{7}$$

The wiggly equal sign $\sim$ is used to equate things which are not equal at all...

Equation (7) is the structure of the Christoffel symbol. It contains one term which is the power one of h. And this is multiplied by the first derivative of h.

We are also going to assume that ∂h is small. This would make a mathematician jump. But physically it means that, far away from the pulsar, not only the ripple h itself is small, in the sense that it has a small amplitude, but that its variations are attenuated too. So the derivative of h is also considered to be a small thing.

η times derivative of h is one order of magnitude small. h times the derivative of h is two orders of magnitude small. It is quadratic in the fluctuation, or in the small gravitational field, so we ignore it. For instance, if $h = .01$ and $\partial h = .01$, then their product is .0001.

In this approximation procedure $\eta\partial h$ is just ∂h. So from equation (7) we deduce that, in the approximation of weak gravitational radiation, the Christoffel symbol is simply proportional to some collection of derivatives of h.

What about the Riemann tensor then ? It will contain derivatives of Γ and products of two Γ's. That will be various kinds of $\partial^2 h$, and various products of the form $\partial h \partial h$, that is second derivatives of the gravitational field and products of first derivatives of the gravitational field. Incidentally, h is called the *gravitational field*, or the *field of gravitational waves.*

$$\mathcal{R} = \partial^2 h + \partial h \partial h \tag{8}$$

So immediately, we say that $\partial h \partial h$ is an order of magnitude smaller than $\partial^2 h$ and again we ignore it.

Finally the Ricci tensor has the same form as the curvature tensor. So whatever the devil the Ricci tensor $\mathcal{R}_{\mu\nu}$ is, it is composed out of simple second derivatives of the metric tensor.

From that we can conclude that Einstein equations have a relatively simple form. There is still plenty of indices around, and the number of different terms involving second derivatives of $h_{\mu\nu}$ is significant. It is complicated enough. So we don't want to write it out here, but it is basically built out of second derivatives – second derivatives with respect to position, second derivatives with respect to time, and maybe even some terms combining a derivative with respect to position and a derivative respect to time.

Furthermore there are several components of it How many components of it ? Well, the Ricci tensor has a μ and a ν, each running from 0 to 3. That makes 16 components. But the whole works, whatever it is, is composed of second derivatives of h.

So our equation of motion is some kind of equation that looks like this

$$\partial^2 h = 0 \tag{9}$$

This leads us naturally to the next topic.

Gravitational waves

Equations like equation (9), especially in relativity, are usually wave equations. Let's recall what the wave equation for an ordinary wave looks like. Let's say we look at a wave that is moving along the z-axis as time goes by. So consider a wave field that we can call $\phi(t,\ z)$. It will satisfy

$$\frac{\partial^2 \phi}{dt^2} = \frac{\partial^2 \phi}{dz^2} \tag{10}$$

This is the simplest wave equation that you can imagine. And simple solutions to it are waves which move either to the right or to the left, figure 1.

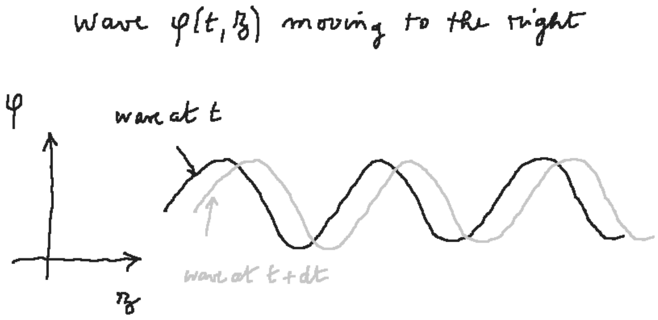

Figure 1 : Wave moving to the right.

You can add a wave which goes to the right and a wave which goes to the left. It is still a solution because the equation is linear.

If we had more directions of space the structure of the equation would be a little more complicated. Instead of just derivative of ϕ with respect to z we would also have derivatives with respect to x and to y. The equation could be rewritten

$$\frac{\partial^2 \phi}{dt^2} - \frac{\partial^2 \phi}{dx^2} - \frac{\partial^2 \phi}{dy^2} - \frac{\partial^2 \phi}{dz^2} = 0 \tag{11}$$

But it is evident that there is a family similarity between the kind of equation like (9), and the kind of equation like (11). And in fact by clever manipulation you can make these equations look exactly alike.

Incidentally equation (9) is not just one equation. There is an equation for each μ and ν. Let's put brackets around it to remind ourselves that there are sixteen components of $\mathcal{R}_{\mu\nu}$:

$$\left[\partial^2 h\right]_{\mu\nu} = 0 \tag{12}$$

These are not all independent. It is really quite a bit simpler than that. But still in principle there are 16 components, therefore 16 equations.

The collection of equations (12) is somewhat similar to Maxwell's equations in that respect. Maxwell's equations have the form of wave equations, for example for the electric and magnetic field. But there are several components. There are three components of electric field, three components of magnetic field. It sounds like there is only six equations but in fact there are eight equations in the complete set of Maxwell's equations.

It is the same sort of pattern with equations (12) : several equations, but all of similar form, and not all independent.

Pay attention to the fact that equations (12) are only suggestive. But we proved that they involve only second derivatives. In the process of producing equations (12) anything that involves h, or derivative of h or second derivative of h, is small, so *when it is multiplied by another small quantity* (as is the case when we do contraction, which introduces multiplicative terms of the form $\eta + h$) we omit the result because it becomes of a higher order of magnitude in smallness.

Remember that what we do is an approximation. But it is a well-defined approximation. Technically, it is the *linearization of Einstein's equations*. It simply means that in Einstein field equations we throw away everything of higher power that h. It is a good approximation when the gravitational radiation, or the gravitational wave is weak.

So we have equations of form $\left[\partial^2 h\right]_{\mu\nu} = 0$, which we are not going to specifically write down because they are kind of messy.

Before we discuss their solutions, let's go back to the observation we made earlier – and in fact we repeated many times since the beginning of the book – that for the space-time to be flat, the metric doesn't have to be necessarily η. But with a proper choice of coordinates it *can be made to be* η.

That means that there are other ways to represent flat space-time. We can make a coordinate transformation on $\eta_{\mu\nu}$. The metric will change, as well as the form of the solution to Einstein field equation, but it will still be exactly the same physical solution.

Therefore there must be solutions of equations (12) which look like they are non trivial. But they really represent flat space, in coordinates, however, with little ripples in them. So let's do that in Euclidean geometry to understand this point.

Let's consider the flat page in Euclidean geometry. It is a flat space. So one of its metrics is simply δ_{mn}, where m stands for the X-axis, and n for the Y-axis. So we start with "good" coordinates X and Y. Then we introduce coordinates X' and Y' which incorporate little disturbances. And we express the coordinates $(X,\ Y)$ in the coordinates $(X',\ Y')$

$$\begin{aligned} X &= X' + f(X',\ Y') \\ Y &= Y' + g(X',\ Y') \end{aligned} \tag{13}$$

It is a coordinate change. We haven't changed the space in any way. All we have done is change coordinates. The new slightly wiggly coordinates represented in the old "good" ones are shown in figure 2.

The metric in the "good" coordinates is

$$dS^2 = dX^2 + dY^2 \tag{14}$$

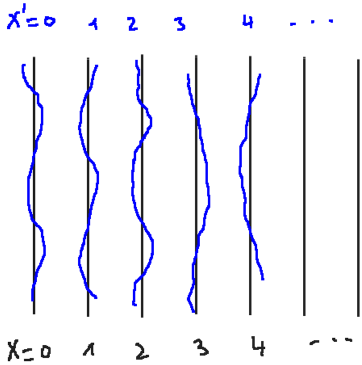

Figure 2 : Flat space with "good" coordinates and slightly wriggly coordinates.

Now what is the metric in the (X', Y') coordinates ? Remember, we are just in a Euclidean space, we are not doing space-time. So let's work out, just for fun, the metric in (X', Y'). We can write

$$dX = dX' + \frac{\partial f}{\partial X'^{\ m}} dX'^{\ m} \tag{15a}$$

likewise

$$dY = dY' + \frac{\partial g}{\partial X'^{\ m}} dX'^{\ m} \tag{15b}$$

Our notations are a little awkward, because $X'^{\ m}$ represents the collection $X'^{\ 1}$, $X'^{\ 2}$, that is simply X' and Y'. And equations (15) incorporate a sum with the summation convention.

Then we plug equations (15) into equation (14). We find out that dS^2 is not $(dX')^2 + (dY')^2$. It contains $(dX')^2 + (dY')^2$

plus some cross-terms. If you work it out – assuming that f and g, and their derivatives, are small so that you can make the appropriate approximations – you will find that

$$dX^2 + dY^2 = (dX')^2 + (dY')^2 + h_{mn}\, dX'^{\,m}\, dX'^{\,n} \quad (16)$$

The right hand side is the square of the infinitesimal distance between two points calculated with the metric tensor expressed in the new wiggly coordinates.

Does this small correction to the metric tensor mean that the page is not flat anymore? Of course not. It just means that the coordinates we use have wiggles in them (when we represent them in "good" coordinates, fig. 2).

It is a nice little exercise to compute, in equation (16), what the correction h is. You will get

$$h_{mn} = \frac{\partial f}{\partial Y'} + \frac{\partial g}{\partial X'} \quad (17)$$

That will mean nothing to you until you try to work out an example, or until you try to prove it.

It is a small perturbation on the metric. It has the form that we already met in equation (4). But it doesn't represent anything physical. It is just a somewhat trivial change of coordinates.

In other words, there exists perturbations on the metric which don't correspond to any physical effect on the space-time – it can remain flat even though the metric has the form of equation (4). Once again a flat space-time is one not where the metric is necessarily η but where *we can find* coordinates such that the metric in those coordinates becomes η.

Likewise, in equations (12), there are small perturbations we can write down which just represent curvilinear coordinates of space-time but don't change its geometry or its physics.

How can we eliminate those phony solutions which automatically solve the equation because they are just flat space-time, in curvilinear coordinates, and don't represent any real physics ?

You do it by imposing more equations on the metric. I could write down the equations but it is not important. What is important is that they will wipe out the unwanted spurious solutions, the ambiguities on the coordinates, in other words the unphysical meaningless solutions of Einstein's equations. And you can do that in a wide variety of ways.

Once you have done that the equations become pretty simple. They become wave equations. The components of h then satisfy perfectly ordinary wave equations, like equation (11). It becomes

$$\frac{\partial^2 h_{\mu\nu}}{dt^2} - \frac{\partial^2 h_{\mu\nu}}{dx^2} - \frac{\partial^2 h_{\mu\nu}}{dy^2} - \frac{\partial^2 h_{\mu\nu}}{dz^2} = 0 \tag{18}$$

Each component of the fluctuation satisfies a wave equation. It means that all the components of the metric just move down the axis so like waves, linear waves.

On the other hand there are also some constraints. There are more equations than one for each μ and ν. The reason is that we have these extra equations to eliminate the spurious fake solutions, that we saw because of the coordinate ambiguity. Once we do that, we find out the physical solutions, the ones that really have meaning. We will classify them.

Suppose we have a wave moving down the z-axis. Here is its equation

$$\frac{\partial^2 h_{\mu\nu}}{dt^2} - \frac{\partial^2 h_{\mu\nu}}{dz^2} = 0 \tag{19}$$

What does it look like ? The simplest solution has the form

$$\phi = \sin k(z - t) \tag{20}$$

It is a wave that at a fixed instant of time is just a sine wave, see figure 1. And it moves to the right down the axis with unit velocity. Unit velocity here means the speed of light. k is the frequency of the wave, also called the wave number. It is the number of complete oscillations per unit length. And of course it can be any number. Short wave length have large k, and long wave length have small k.

Each component of h will have a solution like that. It will be proportional to $\sin k(z - t)$. So let's write

$$h_{\mu\nu}(t,\ z) = h^0_{\mu\nu} \sin k(z - t) \tag{21}$$

The coefficient $h^0_{\mu\nu}$ is not a function of position. It is just a numerical coefficient that multiplies the sine of $k(t - z)$. And it is about all that we can write down. Equation (21) is the nature of a gravitational wave, each component of the metric behaving like a wave moving down the axis.

However, as we said, there are more equations, to filter out spurious solutions. When we impose them all, we find out something interesting. To understand it, let's first of all remember some facts about electromagnetism.

In electromagnetisme we do the same thing : we solve Maxwell's field equations. We can solve them either for the vector potential or for the electric and magnetic fields. But when we write down all the equations, we find some constraints. The constraints are called *transversality of the of the field*. It means that the electric and magnetic fields always point in directions perpendicular to the motion of the wave, figure 3.

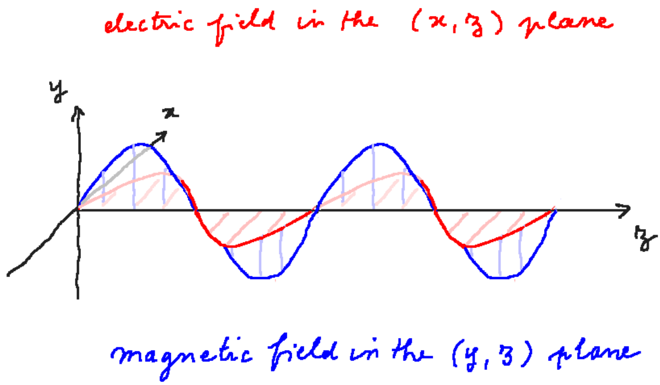

Figure 3 : Electric and magnetic fields photographed at one instant t.

Not only the electric field and magnetic field solutions are waves, but they are transerse waves. That is, the are perpendicular to the axis of progression of the wave, which is the z-axis. And they are perpendicular to each other.

All the little horizontal and vertical sticks shown in figure 3 oscillate over time (not in unisson) in such a way that it suggests a movement down the z axis [4].

Very similar things happen here with gravitational waves. The consequence of the constraints added to filter out spurious solutions is that the waves have to be transverse.

To say that the waves are transverse means that the components $h_{\mu\nu}$ involving t or z are zero. The only components of h which are allowed to be none zero are the components in the plane perpendicular to the direction of the wave.

If we incorporate this fact into the full set of equations – equations (12) plus the equations removing the fake spurious fluctuations –, we find that a gravitational wave has a very simple form. The only non zero components of the perturbation term added in the metric are

$$h_{ij}(t,\ z) = h^0_{ij} \sin k(z - t) \tag{22}$$

where we use the dummy variables i and j, which can each vary over x and y.

Let's think of the spatial part of space-time at time t. And let's slice it into planes at different z, figure 4.

4. When talking about transverse moving waves, usually nothing physically moves in the longitudinal direction. Think for instance of ripples moving on the surface of a water pond. If you look at a cork floating somewhere, it will move up and down as the wave passes it, but it won't move longitudinally.

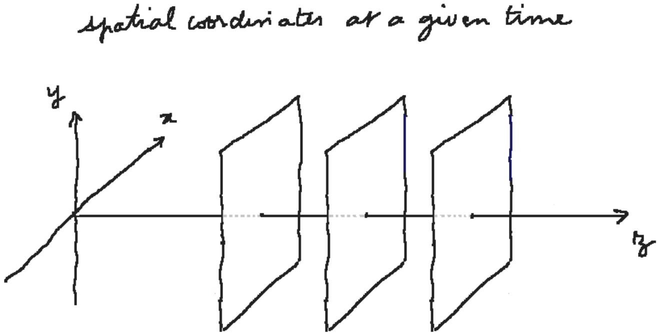

Figure 4 : Slices of x y planes.

At a given location z and given time t the metric h_{ij} is the metric in one of the two-dimensional planes shown in figure 4. The components are simply numbers. At each z and t, the metric is simply a set of numbers.

There is one more equation. It comes from Einstein's field equations. And it says that the trace of the metric h_{ij} is equal to zero. In equation form, this is

$$h_{xx} + h_{yy} = 0 \tag{23}$$

That is it. That is the whole set of equations. And what it tells us is that the metric of a gravitational wave is

$$
\begin{aligned}
h_{00} &= 0 \\
h_{0x} &= 0 \\
h_{0y} &= 0 \\
h_{0z} &= 0 \\
h_{zx} &= 0 \\
h_{zy} &= 0 \\
h_{zz} &= 0
\end{aligned}
$$

That is to say, any component involving either t or z is equal to zero. The only components which are non zero are

$$
\begin{gathered}
h_{xx} \\
h_{yy} \\
h_{xy}
\end{gathered}
$$

They are given by equation (22). Furthermore, from equation (23), h_{xx} and h_{yy} are opposite to each other. The metric tensor is symmetric so for any μ and ν, $h_{\mu\nu} = h_{\nu\mu}$.

That is pretty simple. But what does it mean ?

First of all all of these components of the perturbation term vary with t and z, and only with t and z. If the wave is going down the z-axis, then by definition the variation is along the z-axis.

So h_{xx} is equal to some coefficient h^0_{xx} multiplied by the sine of $k(z - t)$. We can write the formulas for the three non zero terms.

$$\begin{aligned} h_{xx} &= h^0_{xx} \sin k(z-t) \\ h_{yy} &= h^0_{yy} \sin k(z-t) = -h_{xx} \\ h_{xy} &= h^0_{xy} \sin k(z-t) \end{aligned} \tag{24}$$

Let's look at it at a fixed instant of time, for example time t equal 0. As we move down the z-axis, there is a perturbation on the metric. The metric tensor is a little bit different from just a flat space metric – and it is a real curvature not just a coordinate effect. It oscillates as we go down the z-axis, figure 4.

The components that oscillate are the components that have to do with the metric of the plane perpendicular to the wave.

What does it mean to have an h_{xx} added to η_{xx}, which has value one in the most natural coordinates. We write it

$$g_{xx} = 1 + h^0_{xx} \sin k(z-t) \tag{25}$$

It means that proper distances along the x-axis are a little bit different than what they would be if there was no perturbation term. Suppose we are talking about the distance between the point $x = -1/2$ and the point $x = +1/2$ (and $y = 0$). The proper distance is the *real distance* between the two points. Remember, the coordinates x's are just labels. The proper distance is given by the metric. And this proper distance may be either a little bit more than 1 meter or a little bit less than 1 meter, if we are working in meters. That depends on where we are on the z-axis.

For instance, if we are at a point z where $h^0_{xx} \sin kz$ is positive (remember, $t = 0$), that would mean that a stick of length 1 meter would be a little bit shorter than the distance between the two points. It would not be because the meter stick would be shorter – a meter stick is a meter stick –, but because the plane would be stretched in the x direction.

Now what about the other dimension, at the same position z (and the same time $t = 0$) ? We have

$$g_{yy} = 1 + h^0_{yy} \sin k(z - t) \tag{26}$$

But h_{yy} is minus h_{xx}, from equation (23). So we can write

$$g_{yy} = 1 - h^0_{xx} \sin k(z - t) \tag{27}$$

In other words, at a point z where the x direction is stretched, the y direction is compressed. A meter stick oriented along the y-axis would exceed a little bit the distance between the points with y coordinates $1/2$ and $-1/2$. And conversely, when x is compressed, y is stretched. This is shown in figure 5.

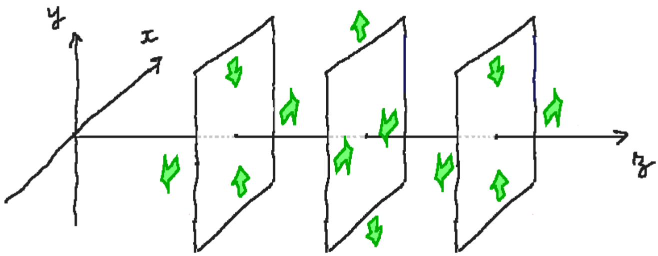

Figure 5 : Alternative stretching and compression.

We can also reason at one fixed point z and see what happens over time. As time flows, the wave passes us. So alternatively our x-axis is stretched or compressed, and our y-axis is compressed or stretched.

This is a real physical effect. The gravitational wave creates curvature. The wave is, if you like, a ripple of curvature of space-time moving with the speed of light. At a given point on the z-axis, when the wave passes it, that is when time flows, it creates a kind of tidal force there. The nature of the tidal force is to actually cause a meter stick in the plane at that position to be compressed and stretched. When it is compressed horizontally, it is stretched vertically, and when it is compressed vertically, it is stretched horizontally.

On the other hand if, at a given time, we look at all the points along the axis, we also see this alternation of compression and stretching, figure 5.

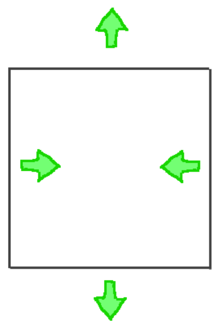

Figure 6 : Tidal forces on a piece of plywood.

At a given point, if you took a piece of plywood, figure 6, you would see it alternatively be compressed horizontally and stretched vertically, and stretched horizontally and compressed vertically.

It is interesting to observe that there is also another solution. Then it is not h_{xx} and h_{yy} which are non-zero, but h_{xy}.

$$h_{xy} = h^0_{xy} \sin k(z - t) \tag{28}$$

And h_{yx} is the opposite.

The metric would look like

$$\delta_{xy} + \begin{pmatrix} 0 & h_{xy} \\ -h_{xy} & 0 \end{pmatrix} \tag{29}$$

where δ_{xy} is the Kronecker symbol, or equivalently the unit matrix. It still has $h_{xx} + h_{yy} = 0$, because both are zero.

It is the same phenomenon as in figure 6, except that the oscillating squeezing and stretching are along the 45° axes. It is not a new solution. It is just the initial solution rotated by 45°. Then any linear superpostion of these two solutions is also a solution.

That is all the gravitational waves there are. You first pick the longitudinal direction in which the wave is moving, then you pick a set of perpendicular axes in the perpendicular plane, and you construct the wave alternatively strectching and compressing as we described.

Questions / answers session

Q. : How can we measure this squeezing and compressing ?

A. : We can use a strain gauge. The wave will create honest real stresses in the piece of plywood. A strain gauge will register them when the wave is going past.

If the wave were static instead of moving, then the piece of plywood would really be unaffected. Everything would be simultaneously squeezed, or stretched, the same way, the rulers which measure the plywood, the strain gauges which measure the stress, etc. But it is the oscillating character of the solution which really does create real honest stresses and strains in it. It does have real curvature.

Q. : What triggers the gravitational wave ?

A. : In electromagnetism a moving charge, for example an electron rotating around a proton, or a current alternating in an antenna, creates electromagnetic waves. Similarly in gravitation, a moving mass will create a gravitational wave, for instance a star or a black hole or a pulsar or whatever, in orbit around another one.

For example the famous binary pulsar of Hulse and Taylor, discovered in 1974, is made of two very concentrated masses, which rotate around each other at a rather small distance. Neither one is a black hole. The whole thing is not a black hole. But it has a very strong varying gravitational field and it produces gravitational waves.

Now near the binary pulsar, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is a bad approximation. The gravitational field is too strong to linearize the equation this way. But if you get at some distance away, then the wave spreads out, dilutes itself and gets weak enough that it becomes a good approximation.

If you are far away, and you take the z-axis to be the line between you and the pulsar, you will feel a gravitational wave passing you, which has the expression (25). There will be stresses and strains in the plane perpendicular to the line of sight to the pulsar.

Q. : How to detect deformation if the meter sticks are deformed too ?

A. : Meter sticks made of matter would indeed be affected. But a wooden meter stick and a steel meter stick would probably react differently. So in theory that is one way to detect the tidal forces created by the wave.

Another way to detect the gravitational wave is to create some system which has a resonance at the frequency of the gravitational wave. A system usually has its own natural frequency of oscillation. If it is reinforced by an oscillating force of the same frequency, bringing in some energy to compensate that lost in the vibrations, it enters into a sustained or so-called driven resonance [5]. Under those circumstances the response will be particularly big.

Notice that the gravitational waves we expect from various souces in astronomy are extremely weak. The additional

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5. See for instance the famous example of the Tacoma bridge, https ://www.youtube.com/watch ?v=3mclp9QmCGs

terms $h_{\mu\nu}$ are extremely small, so small that the dimensionless effect on a rod would be something like 10^{-21}. The fact that people even contemplate measuring them is quite astonishing.

However they can be measured. That is the purpose of the LIGO (Laser Interferometer Gravitational-Wave Observatory) experiments conducted by Caltech and MIT [6] in Livingston, Louisiana, and Hanford, state of Washington. LIGO is a gravitational wave detector. It is not a steel rod. It is a pair of mirrors, plus laser beams, like a modern version of the Fabry-Perot interferometer. And an interference effect, due to the relative motion in the x and y directions, is produced and measured. But basically a gravitational detector is a system which is allowed to be set into oscillation.

The gravitational collapse of a black hole or the collision of two black holes happen from time to time in the universe. We can calculate how many such events happen per unit of time, that are within range of the best imagined detectors. I think we could detect, in theory, one black hole collision per year. It is a lot.

The wave moves at the speed of light. Therefore, if we had at our disposal other means of detection of the black hole collision than the gravitational wave produced, we ought to see the signal at the same time as the wave.

Gravitational radiation is a pretty weak effect when you are far back. From the collision of two black holes, it is an enormous effect when you are close to it. It is then much bigger

6. Waves were detected for the first time on September 14, 2015. See https ://www.ligo.caltech.edu/news/ligo20160211

than any other kind of radiation that is emitted. But if the collision is taking place at large cosmological distances, it can be the case that the only way to detect it would be through gravitational radiation. That is why LIGO is interesting : for astronomy it is a new instrument opening a new window on the universe.

Before the LIGO detection of 2015, gravitational waves had never been observed "directly". But they had already been observed "indirectly" through other physical effects. Just like an orbiting system of electric charges emits electromagnetic radiations which carry off energy, masses which are rotating about each other emit waves and lose energy. Their rotation will speed up a little bit due to this loss of energy.

The study of the binary pulsar of Hulse and Taylor – which are two stars going around each other – showed a perfect match between the gravitational waves it should emit, the energy loss, and the rotation acceleration. This last effect can be measured by the change of frequency of the pulsating light received. Gravitational waves is the main thing which causes the energy of the pulsar to decrease.

The orbital period the Hulse-Taylor binary pulsar is 7.75 hours. Since its discovery in 1974, we have observed its period decrease. As of 2016, it had lost a little less than a minute, exactly as predicted.

Q. : The components h_{00}, h_{0x}, h_{0y}, h_{0z} are all zero. Does this mean there is no curvature along the time axis ?

A. : No. There is curvature in the time direction. The first derivatives $\partial h_{mn}/\partial t$ are not necessarily equal to zero. The curvature tensor has a whole bunch of products of such first derivatives and also has second derivatives. So it will have non zero components involving time. But still it is a very special kind of curvature that is not generic.

Q. : Considering that the Einstein equations are non linear, how did we end up with solutions which are linear ?

A. : This is only because we assumed that the discrepancy between flat space-time and the actual (truly non flat) solution to equation (1) was very small. The linearity is because the wave is weak. Generally speaking, small oscillations about an equilibrium, in a first approximation, can be taken to be linear. In the Taylor expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + ...$$

if h is small we can take h^2 and the higher powers of h to be zero. Then we have

$$f(x+h) \approx f(x) + hf'(x)$$

That is the kind of approximation we make, for instance, when we assimilate a swinging pendulum (with small swings) with a mass attached to a spring whose restoring force is an exact linear function of the displacement around the rest position. This is also the approximation we make when we linearize any type of vibration around a static equilibrium.

Q. : How far off from an accurate solution are we ? How much error do we make ?

A. : It depends how big is h. If if h is $1/10$, and we wanted to add terms in h^2, the corrections would be 1%.

Q. : If h is large enough that we have to add higher order terms, will that affect the transversality of the waves ?

A. : No. The solutions will still be transverse, but won't simply add to give new solutions. For instance two waves in opposite direction will no longer go through each other, they will scatter.

Q. : If we compare these gravitational waves to mechanical waves, can we say that their speed depends on some stiffness coefficient ?

A. : The speed of gravitational waves is the speed of light. It is true that for mechanical waves propagating in some material, their speed depends upon the stiffness of the material, what is called its Young [7] modulus. So you can substitute stiffness of support for velocity of propagation. In that sense, that is, in that mechanistic way to understand physical phenomena, inherited from the XVIth and XVIIth centuries, the medium in which light or gravitational waves propagate – what used to be called the *ether* – is the stiffest support possible.

In the old days, in the XIXth century and before, people thought there was an ether. It was some sort of immaterial space, encompassing the whole universe, in which the

7. Thomas Young (1773 - 1829), English polymath who made important contributions in optics (Young slits), solid mechanics, medicine, Egyptology, etc.

notion of absolute location made sense. They thought that waves – for instance, light, after Young had shown it had wave-like properties – moved through the ether as mechanical waves move through a steel bar, or sound moves through the air. In other words, they thought that light was a vibration of the ether, which was propagating very fast. The very high speed of light was explained by the fact that the ether was very stiff. This mysterious substance was at the same time immaterial and the stiffest that existed. It had to have a very big Young coefficient because the speed of light is much higher than the speed of a wave propagating in a steel bar (about 20 times the speed of sound in air).

This way of thinking about the world, even though apparently sensible, lead to all sorts of difficulties and subtle contradictions.

Anyway, it was definitively shattered by the Michelson-Morley experiment (1887) and by Einstein explanations (1905)[8] : the speed of light and all the laws of physics are the same in every inertial reference frame. Speeds don't exactly add up like in Newtonian physics. There is no ether. There is no absolute motionless reference frame. We don't have an absolute position – let alone at the center of God's creation. Time is not universal, etc.

We all have in the mathematical toolbox we carry in our mind a 3D Euclidean space. But we cannot apply it, as would seem natural, to the entire universe. Most surprisingly, it is more like a ruler that we can use locally to measure things.

8. See volume 3 of the collection *The Theoretical Minimum*, devoted to Special Relativity and Electrodynamics.

In recent history every century has brought new disturbing ways of thinking about the world : the spherical Earth floating in space [9], the Sun rather than the Earth at the center of the universe, light travel which is not instantaneous, the relativity of inertial frames, etc., to limit ourselves to a small sample and to stay in the realm of physics. One must be prepared for new scientific revolutions in every domains of knowledge, because scientific revolutions will not stop coming about. Curiously enough, another commonsense idea – the idea that knowledge progresses toward a final universal immutable truth – must also be relinquished.

Scientific revolutions are not new and will not stop, but one thing is new : their speed of renewal.

That is it for gravitational waves. You can learn more about them by any good book on general relativity. They will write down all the details of the solutions to the equations. These will fill a page or two, and won't be terribly illuminating. But we covered the basic principles.

Einstein-Hilbert action for general relativity

To finish this course, we want to present another way of thinking about the equations of general relativity. Again, solving the equations is too complicated, and we won't do it here.

9. As an exception, that one dates from Antiquity.

Nevertheless, the basic idea is easy to grasp. It is to apply to general relativity the most central principle of physics that we have, that is, *the principle of least action.* Indeed, all the systems that we know about, mechanical systems, electromagnetic radiation, quantum field theory, the standard model of particle physics, etc., are governed by an action principle.

In ordinary particle mechanics, the action attached to a trajectory is an integral over time along the trajectory of the particle. We met it in volume 1. In field theory, the action attached to a field is an integral over space and time. A field configuration is a value of the field at every point of space and time. We met it in volume 3.

We also saw that the first case is a special instance of the more general second case. The trajectory of a particle can be seen as a field whose substrate space-time is reduced to simply one time axis, instead of a space-time with spatial and time coordinates, see figure 7.

Not every configuration of the field is a solution of the equations of the theory, just like not every trajectory is a solution of the equations of motion. Every trajectory is a thinkable trajectory, but not every trajectory satisfies the equations of motion. In the same way every configuration or, better yet, every *history*[10], $\phi(t,\ x)$ is a thinkable possible value of the field in space and time. But they are not all solutions.

10. The variable t allows us to view $\phi(t,\ x)$ as the history of a spatial field evolving over time, just like the trajectory of a particle can be viewed as the history of its position.

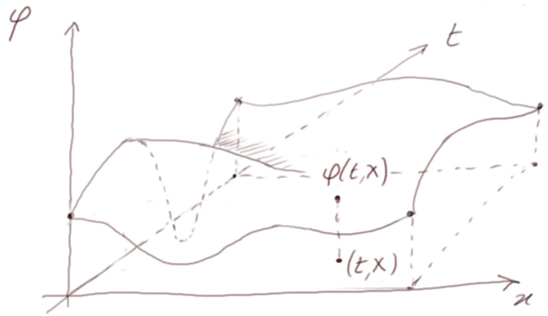

Figure 7 : Field $\phi(t,\ x)$. If we omit the spatial dimension x, we have, in the $\phi\ t$ plane, the trajectory of a particle over time.

In both cases, the behavior of the particle (its actual trajectory), or the behavior of the field (its actual configuration), is that which *minimizes the action*. More accurately, as we stressed many times, it is the configuration which makes the action stationary.

In the case of a particle, the quantity integrated over is called the *lagrangian*. In the simplest situations, it is the kinetic energy minus the potential energy of the system. In the case of a field, the quantity integrated over is called the *lagrange density* (because there is a spatial integration as well as an integration over time).

Let's focus on the case of a field with time and space in its substrate set, that is $\phi(t,\ x)$. And x is a generic variable, which we could also denote X, standing for x, y and z. When we write X^m we mean x, y and z. And when we write X^μ we mean t, x, y and z.

The action attached to this field has the form

$$A = \int dX \ dt \ \mathcal{L}(\phi, \ \phi_\mu) \tag{30}$$

where $\mathcal{L}$ is the lagrange density. It depends on ϕ as well as on its derivatives with respect to the four X^μ, which we denote ϕ_μ.

What do we do with this A? We minimize it, or we make it stationary. We find the field configuration which minimizes (we won't repeat each time "or make it stationary") the action subject to a constraint.

The constraint is that we are given the values of the field on the boundary of a region of space-time. And the minimization process will yield the value of the field in the whole region. This region boundary plays the same role as the initial and final positions of a particle, at time t_1 and time t_2, in the case of particle motion. And the solution of the minimization of the action yielded the whole particle trajectory between t_1 and t_2.

In the case of our field $\phi(t, \ x)$, or $\phi(t, \ X)$, the boundary can be the rectangular region in the x t plane of figure 7, or it can be a more elaborate boundary.

The point of this brief review of what we learned in volumes 1 and 3, about minimizing an action for a particle trajectory or for a field $\phi(t, \ X)$, is that the Einstein field equations can be derived from an action principle. It is a long computation but the action given by equation (30) is simple.

From the action, or more accurately from the lagrange density, we write the Euler-Lagrange partial differential equations in the multidimensional space-time case (as we did in volume 3) which generalize

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = \frac{\partial \mathcal{L}}{\partial X} \tag{31}$$

And after a few hours of tedious but straightforward calculations we land on the Einstein field equations.

Let's look at what the lagrange density is. It has some interesting pieces in it. But first of all let's do a simpler thing : let's review what the space-time volume is and how it is calcultated.

Going one more step back, let's consider first an ordinary space, say a two dimensional Riemannian variety. The index m of the coordinates runs over 1 and 2, and so does n. And when convenient, the first coordinate is simply denoted x and the second y. The variety has a metric

$$dS^2 = g_{mn}\ dX^m\ dX^n \tag{32}$$

The square of the *distance* between two neighboring points is given by this formula (32).

Now suppose we are interested in the *volume* of a small region of space, let's say a small rectangle with sides x to $x + dx$, y to $y + dy$, see figure 8. And suppose the metric has only two diagonal components g_{xx} and g_{yy}. We want to know the area. We used the generic name "volume", but in two dimensions, it is customarily called the area of course.

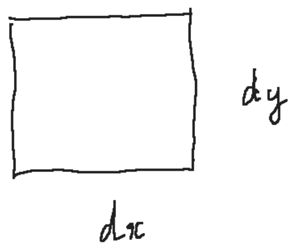

Figure 8 : Small area, also called small "volume".

We would know it if we knew the actual proper distance associated with dx and the actual proper distance associated with dy.

What is the proper distance associated with dx ? It is given by equation (32) : it is $\sqrt{g_{xx}}\ dx$.

So the area, or "volume", of the rectangle is

$$volume = \sqrt{g_{xx}\ g_{yy}}\ dx\ dy \tag{33}$$

The important thing is that it is not just $dx\ dy$. No more so than the proper distance between two neighboring points on the x axis would be just dx. The coordinates are only labels. It is the metric which gives the distances – that is why it is called the metric. It warps everything if you like. Only in Euclidean geometry do the labels correspond directly to distances[11].

We used the case where the metric only has diagonal terms, because it is easier to calculate the volume, and to illustrate the difference between $dx\ dy$ and the volume. In other words, equation (33) is just a special case when the metric has the form

$$g_{mn} = \begin{pmatrix} g_{xx} & 0 \\ 0 & g_{yy} \end{pmatrix} \tag{34}$$

There could be, however, other off diagonal components to the metric. Its general form is

$$g_{mn} = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} \tag{35}$$

In that case, the area, or "volume", is obtained from the *determinant* of the matrix on the right hand side of equation (35). For a 2 by 2 matrix, you remember that the determinant is $g_{xx}g_{yy} - g_{xy}g_{yx}$. For 3 by 3, it is a bit more complicated. It is given by Cramer's rule. For 4 by 4 it is obtained as a linear combination of 3 by 3, etc. But we don't need to know the formulas. The notation for the determinant is $|g|$. And the general formula for the volume in figure 8 is

$$volume = \sqrt{|g|}\ dx\ dy \tag{36}$$

11. If you take a flat rectangular piece of rubber, print on it Euclidean coordinates axes, and then deform the piece of rubber with bumps, throughs, stretching and compressing, the coordinates axes no longer give directly distances, but they remain useful labels to mark points and coordinate axes.

This formula holds in any space curved or not.

If we consider a whole region on the variety, and we want to know its area, or "volume", you guessed how we are going to calculate it. We integrate formula (36). It yields

$$volume = \int_{region} \sqrt{|g|}\ dx\ dy \tag{37}$$

Now if we are talking about the volume of a three-dimensional space, the formula is the straightforward generalization of equation (37), with a dz appearing in the differential element.

$$volume = \int_{3-dim\ region} \sqrt{|g|}\ dx\ dy\ dz \tag{38}$$

If we are in a four-dimensional space-time, it is useful to define a similar concept, which would now involve the fourth component dt as well. The rank two metric tensor now corresponds to a 4 by 4 matrix. In general relativity the metric is not Riemannian but Minkowskian, i.e. its signature is $- + ++$, nevertheless at each point in space-time it has a value and a determinant. And the volume can be defined generalizing again equation (38).

$$volume = \int_{4-dim\ region} \sqrt{|g|}\ dt\ dx\ dy\ dz \tag{39}$$

It is called the *space-time volume*, and it is an interesting quantity. *It is an invariant.* It is the same in every coordinate system. That is an important characteristic of the space-time volume. If you "recoordinatize" the region, that is, if you change coordinates, in some sense the physical space-time volume of any region stays the same.

Other quantities which will be invariant under change of coordinates are based on $|g|$. If we take, for instance, the volume element $\sqrt{|g|}\ dt\ dx\ dy\ dz$ and integrate it multiplied by any scalar $S(t,\ X)$, where X stands for $x\ y\ z$,

$$\int \sqrt{|g|}\ S(t,\ X)\ dt\ dX \tag{40}$$

we also get an invariant quantity which doesn't depend on the coordinates.

Indeed, it is just a sum of little volume elements, each multiplied by the value of a scalar field. Since the scalar is, by definition of a genuine physical scalar value, an invariant, the integrand in formula (40) is invariant. Therefore expression (40) is invariant, even though writing it down explicitely necessitates the use of a set of coordinates.

In relativity theory the action is always supposed to be the same in every coordinate system. Otherwise the laws of physics which came out of it would depend on the coordinate system.

If the laws of physics are supposed to be the same in every coordinate system, one way of ensuring that is to make the action an invariant. So it is natural to take for the action

$$\int dX^4 \sqrt{|g|}\ \text{x}\ \textit{some scalar} \tag{41}$$

where dX^4 stands for the complete differential element.

Let's take the Einstein theory without any energy-momentum tensor. No souces, just a pure gravitational field, nothing else in space-time. What kind of scalar can we introduce in expression (41) ? How many scalars are there that we could make up only out of the metric ?

There is for instance the *curvature scalar*. Anything else ? Answer : well, simply any number, 7, 4, or 16, or any number you like. So we can use for the scalar

$$S(t,\ X) = \mathcal{R} + a\ number$$

because numerical numbers are always regarded as invariant. If you see the number 7 at some point in space-time (not the component of something, but a pure number), everybody else sees the same number 7.

The number we add next to the curvature scalar depends on the laws of physics. It is itself a law of physics. It has a standard notation : it is denoted Λ (read lambda). And the formula for the action is

$$\int dX^4 \sqrt{|g|}\ \ [\mathcal{R} + \Lambda] \tag{42}$$

This additional number Λ is called the *cosmological constant.* It creates a term in the lagrange density which is just proportional to the volume itself.

In the last chapter, we did not introduce the cosmological constant. We just discussed it briefly in the questions / answers section of that chapter. If we don't introduce it, the action reduces to

$$\int dX^4 \sqrt{|g|}\ \mathcal{R} \tag{43}$$

Einstein's field equations are what we obtain when we minimize expression (43), or make it stationary. Therefore at the solution we must have

$$\delta \int dX^4 \sqrt{|g|}\ \mathcal{R} = 0 \tag{44}$$

Question : Where would the energy-momentum tensor appear in the lagrange density $\sqrt{|g|}\ \mathcal{R}$?

Answer : the energy-momentum tensor would come in as additional terms in the lagrange density which would depend on the material and other fields. For example there could be electromagnetic energy. There could be other kinds of energy. They would come in as additional terms under the integral sign in equation (44).

Equation (44) is the vacuum case. It is what governs gravitational waves.

Other things could indeed enter into the lagrange density, to start with, sources of course. But they could also be made up from different fields. However, the lagrange density will always have the square root of determinant of g in factor. Its general form will be

$$\sqrt{|g|}\ \ [\ \mathcal{R} + \textit{other terms}\ ] \tag{45}$$

For instance, if you remember your electromagnetism, the electromagnetic field is described by an anti symmetric tensor $F_{\mu\nu}$. Expression (45) would then be

$$\sqrt{|g|}\ \ [\ \mathcal{R} + F_{\mu\nu}\ F^{\mu\nu}\ ] \tag{46}$$

That would govern the electromagnetic field.

But let's leave that out. And let's just finish with the Hilbert-Einstein action for the vacuum case

$$\mathcal{A} = \int dX^4 \sqrt{|g|}\ \mathcal{R} \tag{47}$$

It was discovered, I think, just about simultaneously by Einstein and Hilbert. But Einstein already had the field equations which he had derived following the method presented in chapter 9 :

$$\mathcal{R}^{\mu\nu} - \frac{1}{2}\ g^{\mu\nu}\ \mathcal{R} = 8\pi G\ T^{\mu\nu} \tag{48}$$

He already had the whole works, and knew how it all fitted together.

Hilbert and Einstein independently sat down and asked themselves : is it possible to derive equations (48) from an action principle ?

Both came up with the same answer that it is. They derived equations (48) from the action given by an extension of equation (47) when there are sources in space-time, that is, when elements from the energy-momentum tensor complete the lagrange density.

So equation (44) is a physical principle that contains all the Einstein field equations.

To clarify further what we said in this section about fields and applying the principle of least action to them, what is the field $\phi(t,\ X)$ in general relativity, which appears in equation (30) ? Answer : it is the metric.

$$\phi(t,\ X) = g_{\mu\nu}(t,\ X) \tag{49}$$

Equation (47) describes an action that is made up out of a metric.

The *principle* to find the actual metric created by the physics is to vary the metric a little bit in equation (47) and look for that which minimizes the action.

If we move away from the solution g, the action given by equation (47) should grow bigger than at the solution.

The *mathematics* is to solve the Euler-Lagrange equation for a field

$$\sum_\mu \frac{\partial}{\partial X^\mu} \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial X^\mu}} = \frac{\partial \mathcal{L}}{\partial \phi} \tag{50}$$

where $\mathcal{L}(\phi,\ \phi_\mu) = \sqrt{|g|}\ \mathcal{R}$. And where $\mathcal{R}$ is itself a function of g.

In conclusion, the Einstein-Hilbert form of general relativity is equation (44) which we reproduce below

$$\boxed{\delta \int dX^4 \sqrt{|g|}\ \mathcal{R} = 0}$$

Steps to derive the field equations from the action

Here is the complete procedure, starting from the last equation, to arrive at Einstein field equations :

1. Express the Christoffel symbols in terms of the metric. It involves first derivatives of the metric components.
2. Express the curvature tensor in terms of the Christoffel symbols. It involves first derivatives of the Christoffel symbols and products of Christoffel symbols.
3. Express the curvature scalar by contracting the curvature tensor.
4. Multiply it by the square root of the determinant of the metric. It is a matrix of four rows and four columns. The determinant is a great big thing made up out of lots of components of the metric.
5. Apply the Euler-Lagrange variational equations (50) to the lagrange density that you wrote.

After a few hours of calculations, you will end up with Einstein field equations.

We are finished with general relativity. We hope that you enjoyed it, and look forward to seeing you in the next courses on cosmology (notes available on the site of the notetaker at `https://www.lapasserelle.com/cosmology`) and statistical mechanics (notes available at `https://www.lapasserelle.com/statistical_mechanics`).