Lesson 3 : Flatness and curvature

Notes from Prof. Susskind video lectures publicly available on YouTube

Introduction

General relativity has a reputation for being very difficult. I think the reason is that it is very difficult. It is calculation-intensive : symbols, indices, awe-inspiring equations. There are ways people have invented to express things in more condensed very neat notations. But just learning them in itself is a task that that would take some time. Things like deerbines (?), forms, spinors and twistors and all sorts of other mathematical objects. You could call many of them just notational devices if you like. And they do simplify the equations.

So I sometimes feel that in presenting these things the way I do, it is sort of like Maxwell who wrote down every single equation of his set of Maxwell equations. At first he wrote twenty altogether. Although now we write only four.

We don't usually write all components of the equations. We put them together into vector notation and so forth. And if we are smart, we can even avoid the indices by inventing symbols like del, curl, laplacian, etc. The same thing could be done to some extent for general relativity. But as I said the computational techniques are harder.

I am by no means an expert in general relativity. I know the subject, how to use it. But when I have to do something involving general relativity, I go back to the indices. I can follow if somebody else uses the fancy methods, but I can't get around to doing them myself. It is not that I don't know how to use them. But they are not natural to me. I wind up going back to the same things we are doing here : indices, derivatives, the whole works.

My feeling is that it is the right way to see it for the first time. But it is a nuisance. It has indices all over the place, notations all over the place, too much of it. We can't help it. So some aspect of it you have to get over. You have to get over that hump of tensor notation, bunches of indices upstairs and downstairs. And then it goes easier, more smoothly... until you come to do a calculation. Then you are right back into the mess again.

Now if you are really interested in doing the computations in general relativity, there are packages. You just put in the metric as a function of position, the metric tensor. And out it will spit various tensors, Riemann tensors, this kind of tensor, that kind of tensor, Einstein tensors. And then you can say, without even looking at the results : okay, please Mr. Computer, set the Einstein tensor equal to the energy momentum tensor and tell me what comes out. So, yes, computers can do a lot better than us.

As I said, general relativity is hard. And part of the difficulty is just the complexity of the notations that are necessary. By the way, the subject that we have been studying up to now is Riemannian geometry. It is the geometry of smooth curved metric spaces. They can have highly non-flat shapes, but locally they are like planes or higher dimension flat spaces, and their local geometry is like Euclidean geometry.

I suppose we have Riemann to blame for the complexity of the notation. Although, actually he simplified it. Einstein simplified it further. But nevertheless it is notationally and computationally intensive.

The basic concepts are really not so hard. But we have to go through the mess before we get to the concepts or before we can express those concepts.

General relativity in modern physics

One of the most active areas of research among theoretical physicists, at the beginning of the XXIst century, is the quantization of general relativity, the quantization of black hole physics. The other most active area is quantum field theory.

There are also astrophysical areas where black holes are very important. Astrophysical areas are entirely classical general relativity.

There is numerical relativity. It means actual computations with numbers. This involves special mathematical techniques employed with computers. It is usually used for the purpose of solving astrophysical problems : putting in numerically various initial situations, for instance a pair of black holes orbiting around each other, and simply running the computer to find out what happens, by solving Einstein's equations.

It is a very subtle business by the way. You don't just put in the equations and leave the computer to grind. It is much more subtle than that. Numerical relativity is also a very hard subject. The purpose for instance can be to find out how much gravitational radiations are emitted when two black holes fall into each other. These are entirely classical radiations. Bang ! A lot of gravitational radiation is produced, gravitational waves.

Why are we interested in them ? Among other reasons, because with the latest detectors we have a chance of detecting them. Anything you can detect is a window onto a new natural phenomenon. So gravitational wave detectors may be able to detect gravitational waves from the collision of two black holes. It is estimated that there are every year a few collisions of black holes that can be seen by their gravitational radiation.

So there is a lot of different active areas nowadays related to general relativity – classical general relativity, that is involving no quantum uncertainty. The equations of general relativity are still full of things to teach us.

Furthermore the equations of black holes have been found to be extremely useful in condensed matter physics and fluid dynamics, in other words in domains having nothing to do with black holes and gravity. Different domains, same equations. In short, there is a heavy amount of activity devoted to applying the equations of general relativity to other areas of physics. It is probably one of the most active area of research.

This preeminence of general relativity has not always been the case. Over the career of Professor Susskind in physics it went from being real backwaters in physics to the forefront. Nobody studied general relativity very much in the 1960s when he was a student, except people who were really interested in the subject per se. They were then focused specifically on questions of general relativity. But the bulk of theoretical physicists were not interested.

Then it started to change at the end of the XXth century. It became interesting in astrophysics and eventually took over questions of fundamental theoretical physics. Why? Because of a clash of gravity in general relativity [1]. Anyway, for whatever reason, it is probably now more important than the day-to-day workings of theoretical physicists in almost any other subject – that and quantum field theory, and the connection between them.

1. The equations lead to the hypothesis of black matter (see volume 5 of the collection *The Theoretical Minimum*, on Cosmology) just like problems related to light lead to special and general relativity.

Riemannian geometry

This is the last chapter in which we will be studying Riemannian geometry as such, without really discussing gravity. In the next chapter we will really get into gravity. What do all these manipulations of tensors have to do with gravity ? We saw it a little bit in the first lesson.

The problem of finding out whether there is a real gravitational field, as opposed to just some funny coordinates with which we are locating events in space-time, leading to curved lines but *not intrinsically* curved trajectories (see figure 4 of chapter 1), is mathematically identical to the problem of finding out if a certain geometry – characterized by its metric tensor – is flat or not.

Let's think of a two dimensional space or variety to begin with. That means a surface $\mathbb{S}$ of some sort, where each point is located with two real coordinates X^1 and X^2. See figure 1, where it is shown embedded in the usual 3D space for convenience :

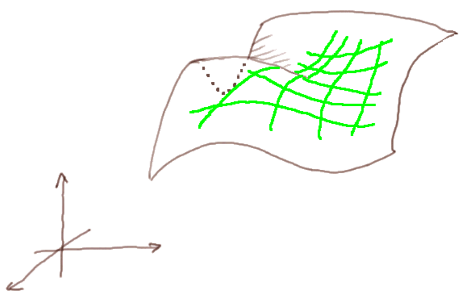

Figure 1 : Two dimensional variety
with a system of coordinates on it.

For the system of coordinates, mathematicians technically talk of a mapping from $\mathbb{R}^2$ into $\mathbb{S}$, with nice properties of smoothness and differentiability.

And $\mathbb{S}$ has a metric, defined at any point P with coordinates X on it, by the now familiar equation

$$dS^2 = g_{mn}(X)\ dX^m\ dX^n \tag{1}$$

where m and n run over 1 and 2.

Notice that this metric defines the distance between two infinitesimally close points, not between two points far away.

A flat geometry is one for which Euclid's postulates are correct. There are parallel lines, all the stuff we learned in high school Euclidean geometry. And the surface $\mathbb{S}$ can be laid out on a plane without applying any distorsion, stretching or compression, on it. In the case of two dimensions, such a surface is also called a developable surface.

Any geometry is characterized by its metric tensor. A flat one is not necessarily one whose metric, given by equation (1), has the special form of the Kronecker delta tensor. That is not the right statement. A flat geometry is one where *we can find another system* of coordinates Y such that at any point P, now located with those Y coordinates, the metric has a simple form

$$dS^2 = (dY^1)^2 + (dY^2)^2 \tag{2}$$

Notice that to find such a transformation is always possible locally at any *given* point P, because locally any surface is like a plane. But it is not always possible to find such a transformation globally over the whole surface.

In other words, more generally, given an arbitrary metric tensor which varies from place to place – assuming it is smooth, that it has all the good differential properties and all – the problem of determining whether the corresponding space is flat is a somewhat difficult problem.

The bad way to approach the problem is to search through all possible coordinate systems, that is for each possible coordinates Y to transform the initial metric tensor and see whether its new form is the Kronecker tensor. Of course, this will take an infinite amount of time. We will never get there.

So we need a better technique. The better technique is to search for a *diagnostic quantity*, that is a quantity which is built out of the metric and its derivatives, that we can calculate [2]. If it is zero everywhere throughout the space that will say the space is flat. If it is non-zero at some place, it will say that the space is curved there.

The diagnostic quantity that does the job is called the *curvature tensor*.

The goal of this chapter is to get to the curvature tensor of a Riemannian geometry. It is a complicated goal – I warn you about this. You will have to pay attention or else you will fall asleep. It can be a very soporific subject, so we will try from time to time to tell a joke to keep you awake. But on the other hand, if you can follow it, you will learn a lot.

2. it will be a quantity computed at every point, so in reality it will be a scalar field.

And it is very interesting. I think you will find it very interesting. We will try not to make it too boring.

What do we start with ? We start with a space. And a space means first of all a number of dimensions. In Riemannian geometry the number of dimensions can be any integer. In principle you can even have a zero-dimensional space. But that is just a point ! There isn't much to say about the geometry of a point. So let's go to the next number of dimensions.

A one dimensional space is either an infinite line or a closed curve – a loop. If it is a closed curve what is it characterized by ? It is characterized by only one thing : the total length of the curve. Every loop is equivalent to every other loop of the same length. In other words – just think about it for a moment – take a piece of rope that closes on itself to form a loop and has a certain length. Wiggle it or curve it in any kind of ways, it can always be mapped or put on top of another piece of rope of exactly the same length. It can be done in a precise way without stretching either of them. A bug crawling successively around the two loops of the same length would not notice any difference.

So one dimensional spaces are all equivalent to each other, up to their length. There are only two general kinds : either they are infinite, or they form a loop of a certain length L. And every loop of length L is identical to any other loop of the same length.

And it is an intrinsic property. The bug walking along the closed rope cannot distinguish the shape we give it in 3D. All it can do is count the number of steps it takes to walk around the rope. For instance, the bug makes an initial mark someplace, goes around the rope till it comes back to the mark, and records the number of steps it took it to do it. That is the only thing the bug can say, or measure, about the closed rope.

Two dimensional spaces are where things start to be more complicated – and more interesting. There are flat ones. There are curved ones. A flat one is a plane. A curved one could be a sphere. It could a space with lumps, the surface of the Earth including the mountains and the valleys.

It can even have a weird topology, for instance the surface of a donut – also called a torus. You can poke another hole in the torus and make a torus with two holes, and so forth. So by the time you get the two dimensional spaces they are pretty complicated. Nevertheless they are characterized by the metric tensor.

We are concerned with whether they are flat. The first question then is : find something which distinguishes whether they are flat. If g_{mn} is not equal to δ_{mn}, that doesn't mean the space is not flat. It just means you may be in the wrong coordinates. You can't diagnose the space by asking whether g_{mn} is δ_{mn}.

We want to find a quantity called R – not necessarily a scalar – such that if it is zero everywhere the space is flat, and if it is not zero someplace the space is not flat there. It would be best if it is a tensor, because if it is a tensor and it is non-zero, it is non-zero in every frame of reference. R stands for Riemann. He was the first person who asked this question and then answered it.

So we want to find a tensor R such that if R is equal to

0, that implies that the space is flat *and* that there exist coordinates such that

$$g_{mn} = \delta_{mn} \tag{3}$$

That is what we are going to search for : the *curvature tensor.*

Recall a few properties of the metric tensor and what you do with it. First of all g_{mn} is by definition a function of position in general : $g_{mn}(X)$. If it were not a function of position, if it was a constant matrix, that is a constant set of coefficients, then it would be very easy to find coordinates in which g_{mn} is δ_{mn}. You simply have to straighten out the angles of the coordinate axes and adjust the units so that they correspond to length. So if g_{mn} is not dependent on position, then the space is definitely flat.

So we start with $g_{mn}(X)$, where X is the set of coordinates of a point P. What do we do with it ? We compute

$$g_{mn}(X)\ dX^m\ dX^n$$

That gives you the square of the length of the little differential displacement dX.

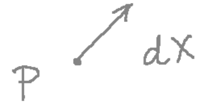

Figure 2 : Little displacement dX.

We call the displacement dX for simplicity, but we could call it dP. It is indeed a small vector which will have different components in different coordinate systems. In a Y system, we will call its components dY. So the square of the length of the little displacement is

$$dS^2 = g_{mn}(X)\ dX^m\ dX^n \tag{4}$$

The metric tensor *is* a tensor. We proved that in the last lesson. We used the fact that dS^2 is an invariant quantity. All observers, that is all coordinate systems, should agree about the length of a given little vector as shown in figure 2. You express the vector in Y coordinates instead of X coordinates. The length has to be the same and that leads immediately to the proof that g_{mn} really is a tensor, that is, transforms like a tensor.

Next property of g_{mn} is that it can be viewed as a matrix and it has an inverse. For instance in four dimensions, the covariant metric tensor g_{mn} is also the symmetric matrix

$$g_{mn} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & g_{22} & g_{23} & g_{24} \\ g_{13} & g_{23} & g_{33} & g_{34} \\ g_{14} & g_{24} & g_{34} & g_{44} \end{pmatrix}$$

The inverse of g_{mn} is denoted g^{mn}. It is a matrix, and it is also a contravariant tensor. It satisfies the following equation

$$g^{mn}\ g_{nr} = \delta^m_r \tag{5}$$

A remark on notations : equation (5) is a matrix equation as well as a tensor equation. As matrix equation it says that

g^{mn} times g_{mn} yields the unit matrix. As a tensor equation it says that the tensor product of g^{mn} with g_{nr}, when it is then contracted along the index n, gives the Kronecker delta tensor.

Up to now, we haven't paid much attention to the placing of the indices of the Kronecker delta symbol, see for instance equation (3) where we wrote both of them downstairs because we equated it to g_{mn}. The point to note is that in matrix algebra, the Kronecker delta symbol is generally denoted δ_{mn}. While in tensor algebra of general relativity or Riemannian geometry it is most of the time denoted $\delta^m_{\ n}$ with one upstairs index and one downstairs index (the downstairs index slightly shifted to the right if we like neat typography) – at least when the Kronecker delta symbol is treated as a tensor.

This is a thing to prove : if you take the Kronecker delta symbol and you transform it pretending it is a tensor of whichever type, you just get back the Kronecker delta symbol of the same type.

There is another important point to mention : in Riemannian geometry, as opposed to Einsteinian geometry, all lengths and their squares are positive. This says something about the properties of the metric tensor : there are no directions along which if you evaluate dS^2 you get anything but a positive number. That is equivalent to a statement about the eigenvalues of g_{mn}. It says that all of the eigenvalues of the matrix are positive.

To be complete, the following statements are equivalent :

1. dS^2, of equation (4), is always positive
2. the quadratic form $g_{mn}(X)\ dX^m\ dX^n$ is positive definite

3. the matrix g_{mn} has only positive eigenvalues
4. the matrix g_{mn} is positive definite

That will change when we go to the theory of real gravity and have to use the Einsteinian geometry.

The metric tensor allows you to do something : it allows you to raise and lower indices. Consider a vector with contravariant components

$$V^m$$

You can put it in correspondence with another vector with covariant components. It is really the same vector. We talked about this a little bit in the last chapter. We saw the difference between contravariant and covariant components [3].

To construct the covariant components of a vector, given its contravariant components, we just multiply it by the covariant metric tensor

$$V_n = V^m \; g_{mn} \tag{6}$$

Likewise, and this is something that needs to be proved, the following holds

$$V^m = V_n \; g^{mn} \tag{7}$$

So if you know the upper indices, that is the contravariant components of a vector V, equation (6) is the way you build its covariant components. And if you know the covariant

3. V^m and V_n denote the same geometric vector V. The V^m's are the components of V in the basis of e_i's, see figure 7 of chapter 2. The components V_n's are defined by equations (11) of chapter 2. The V_n's can also be seen as the components of a dual vector. See vector spaces and their dual in any book of linear algebra.

components of V, equation (7) is the way you build its contravariant components. This requires a bit of proof that is left to the reader.

Gaussian normal coordinates

To what extent can you force the metric to look like a flat metric ? Now we mean not only that the space is flat, but that we are in a coordinate system such that g_{mn} is equal to δ_{mn}. If that is the case, the space is definitely flat – that is the definition of flatness. But to what extent given any space can you find coordinates such g_{mn} might be the Kronecker delta over some limited region.

Here is a theorem : at any given point P on the surface, we can find a system of coordinates which are good in the immediate vicinity of the point, see figure 3. They will not be globally flat. Equation (4) will not reduce over the whole variety to Pythagoras theorem.

But in the immediate vicinity of point P, with a suitable choice of coordinates the metric tensor g_{mn} can be made to be approximately like the Kronecker delta. Such coordinates are called *Gaussian normal coordinates*.

Here is the way we proceed. We position ourselves at point P and we move along any first direction as straight as we possibly can. Later we will learn what is meant by "as straight as we possibly can". It will mean along a geodesic. So you make as straight a curve as you can.

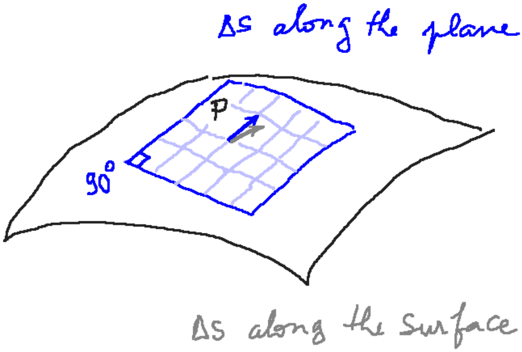

Figure 3 : Displacement of length ΔS along the surface and along the tangent plane in the same direction. The coordinates are represented on the tangent plane. We could also have represented them – slightly curved – on the surface itself.

Say, you are a little bug. You move on the surface following your nose going straight ahead. That defines one coordinate axis. Then you come back to point P. You have some surveying tools to figure out which other directions make a right angle with the fist line. On a two dimensional surface there is only one other direction (in one sense or the other). In three dimensions, there is a whole plane. And you go off in an orthogonal direction as straight as you can. You build that way a complete set of coordinates based on those directions.

The theorem says that at every point P of the surface you can choose Gaussian normal coordinates such that at that point, whose coordinates are, say, X_0

$$g_{mn}(X_0) = \delta_{mn} \tag{8}$$

You can do that in more than one way. If you found coordinates with which equation (8) is true, you can rotate the coordinates. This will produce a different set of axes such that equation (8) in the new set is still true. In figure 3, think of pivoting the coordinate system around P.

The theorem says furthermore that, at point P, once you have chosen the directions, you can also choose the X's such that the *derivative* of any element of the metric tensor $g_{mn}(X)$ at that point with respect to any direction in space, X^r, be set equal to zero

$$\frac{\partial g_{mn}}{\partial X^r} = 0 \tag{9}$$

The proof is actually very simple. It is just a counting problem. You count how many independent variables you have, and how many constraints they must satisfy.

Equation (9) will be true, at a given point, only for the first derivatives. Unless the space is flat, the derivatives of higher order at that point won't be zero

$$\frac{\partial^2 g_{mn}}{\partial X^r \partial X^s} \neq 0 \tag{10}$$

So, at a point, there is no content really in saying that the metric can be chosen to be, let's say, flat-like. Up to the first derivatives included that can always be done.

It is in the second derivatives of the metric tensor that the flatness or non-flatness of the space somehow starts to show up.

How do we prove it ? As said, this is actually not hard. Let's take up the point of interest that we called P, of coordinates X_0, to be the origin.

$$X_0 = 0$$

Now suppose we have some general metric and some coordinates Y in which the metric has some form, which does not satisfies equations (9).

Let's look for some X's which will be functions of the Y's, and choose them in the following way : at the place where $X = 0$, in other words at the origin, let's also assume that $Y = 0$. So the two sets of coordinates have the same origin. That means that X will start out just equal to Y plus something quadratic in Y

$$X^m = Y^m + C^m_{nr} Y^n Y^r \tag{11}$$

plus some more complicated terms. We are simply expanding each X^m in powers of Y^1, Y^2,... Y^N, where N is the number of dimensions of the variety.

How many such C^m_{nr} are there ? Suppose we work in four dimensions. Then there 10 distinct combinations $Y^n Y^r$, because $Y^n Y^r = Y^r Y^n$. And for each n and r we have four C^m_{nr}, when m runs from 1 to 4. That means there are 40 independent coefficients.

Now how many independent components of g are there ? Answer : ten. So there are 40 equations (9).

Finally we reached 40 equations to solve for 40 unknowns. That allows us to be sure, at point P, that not only $g_{mn}(X) = \delta_{mn}$, but also that the derivatives of g_{mn} and δ_{mn} will match up to quadratic order [4]. That means that we will be able to solve the forty equations (9), but that we will fail to set the left hand sides of equations (10) to be equal to zero.

To summarize, at any point P, a smooth variety is locally flat. We can approximate it by its tangent space, see figure 3. And we can construct coordinates X's such that P is located at the origin, and the metric tensor has the form

$$g_{mn}(X) = \delta_{mn} + o(X) \tag{12}$$

where $o(X)$ represent terms of second order and higher. We also interpret equation (12) as saying that the metric is locally Euclidean up to second order.

The reason we introduced this theorem is because we want to give some meaning to derivatives of tensors. We have talked about various operations we can do on tensors to make new tensors : we can add them, subtract them, multiply them, contract them... We already can do all sorts of things.

Can we differentiate them ? In particular, if we have a tensor which is a function of position – that means all of its components are functions of position –, can we construct another tensor which is in some appropriate sense the derivative with respect to position of the tensor that we have ?

The answer is yes. We shall see how to do it and why it is a little bit tricky.

4. It is easy to check that we are in a case where the 40 equations with 40 unknowns do lead to an existing and unique solution. The reader is invited to verify it in two dimensions.

Covariant derivatives

To differentiate a tensor with respect to position, we could think : "Okay. Let's take the components of the tensor – for instance contravariant components – and just differentiate them." That would yield a new set of components, with one more index, which would be simply the derivatives of the components of the tensor. But there is the following problem.

Think for instance of the derivatives of a vector. We could differentiate each component with respect to each direction. We sure can do that. This would produce a two dimensional collection of values. But it would not be a tensor [5]. Let us see why.

Consider a surface and a point P on it, figure 4. We have two sets of coordinates on the surface, coordinates X and coordinates Y.

If the space is flat, for X we just use flat ordinary Cartesian coordinates. Or if the space is curved, we use a set of Gaussian normal coordinates at P, that is coordinates X that are locally, at P, as straight and orthogonal as possible so to speak. And there is another set of coordinates Y, for instance the initial ones. For convenience, at P we chose X such that the X^2-axis is tangent to the Y^2-axis.

5. A tensor is an abstract object which has, in each coordinate system, a multi-indexed collection of components. And these collections of components must transform from one system to another according to equations (14) to (16) of chapter 2 or their obvious generalizations. If we differentiated a vector field, at the same point, using two different coordinate systems, we would obtain two collections of double-indexed values. But they would not be linked by a *tensor equation* like those mentioned, which are necessary to define a tensor.

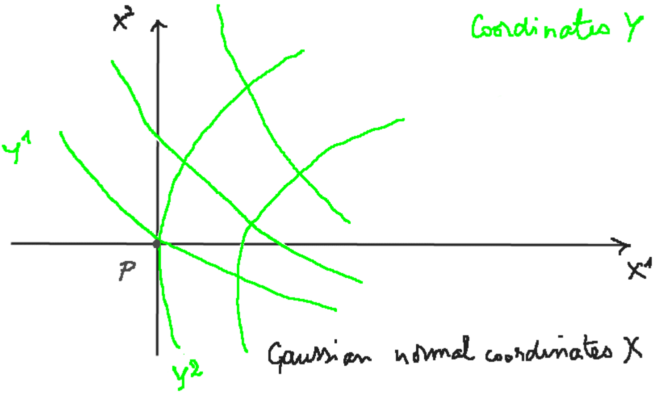

Figure 4 : Surface viewed at P with Gaussian normal coordinates X, and any coordinates Y.

Think of a vector field defined over the surface. The vector field is made of a different vector at every point. In order not to clutter the picture, these vectors are not shown on figure 4 yet. There is one at P and there are plenty around P.

Before we ask how to differentiate a vector field, let's ask what it would mean for the vector field to be constant in space. We run into the following difficulty : because the space is curved it becomes hard to compare the vector at one point with the vector at another point.

The coordinates X cannot be chosen to be everywhere flat. Then what exactly do we mean by saying that a vector at one point is equal to a vector at another point ? It does not really mean anything because to compare a vector at P with a vector at Q, unless we have nice flat coordinates over the whole surface, there is no unique way to do that. So let's look at it for a moment.

If the space is really flat, then we know what it means for a vector at one point P of the surface to be the same or to be equal to a vector at another point Q of the surface, see figure 5. It means they point in the same direction and have the same length. Therefore in the X coordinates they have the same components.

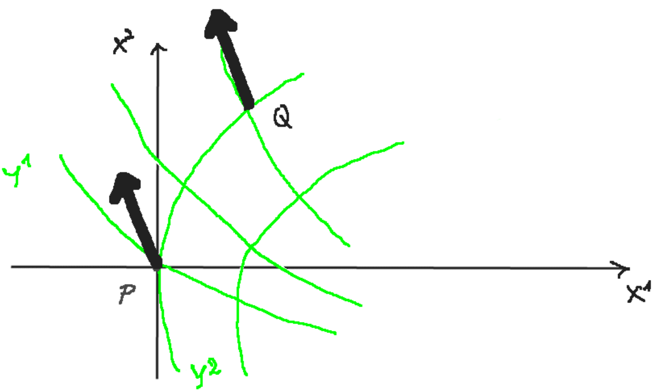

Figure 5 : Two equal vectors, at P and at Q.

But what about their components in the Y coordinate system ? In fact let's take a special case. Suppose both vectors point vertically in the X axes, figure 6. In that case, V_P only has an X^2 component in the flat coordinates. And so does V_Q. They have the same components in the X axes.

Are the components of V_P and V_Q in the Y coordinates the same ? Answer : No. Along the Y axes, V_Q has a Y^1 component and a Y^2 component, while V_P only has a Y^2 component.

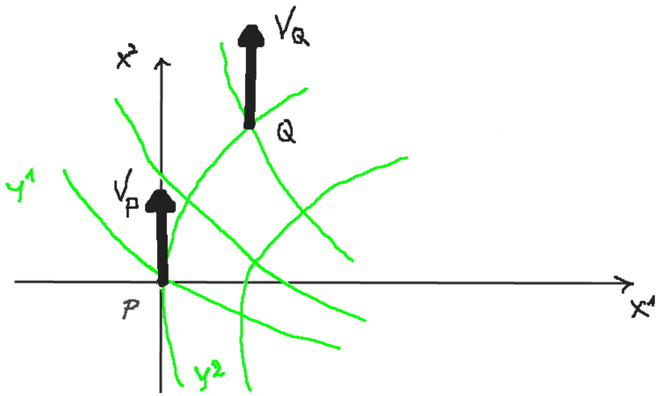

Figure 6 : Two equal vertical vectors, at P and at Q.

So it's clear that even though the vectors V_P and V_Q are the same, their components are not the same in the Y coordinate system.

This will be true, be it for covariant vectors or for contravariant vectors : when we have curvilinear coordinates, we cannot judge the equality of two vectors. And in general we *will* be dealing with curvilinear coordinates.

Another way to say the same thing is that the derivative of the m-th component of V, with respect to the r-th direction of the coordinate system, might 0 in one coordinate system and not 0 in the other coordinate system.

$$\frac{\partial V_m}{\partial X^r} = 0$$

$$\frac{\partial V'_m}{\partial Y^r} \neq 0$$

It might even be the case – as in figure 5 or 6 – that *all of the derivatives* of V_m are 0 in one coordinate system and not 0 in the other coordinate system. That will be because the coordinates are shifting, not because the vector is changing.

You see what we are driving at : the equation saying that the derivatives of a vector are all equal to 0 may true in one reference frame, but untrue in another. Therefore it cannot be a tensor equation. In other words

the ordinary derivatives of the components of a vector with respect to the coordinates do not form themselves a tensor.

If they were components of a tensor we would think of this quantity

$$T_{mr} = \frac{\partial V_m}{\partial X^r}$$

as a rank 2 tensor with an m index and an r index.

But it were a tensor, the following fact would have to be true : if T_{mr} is zero in one frame or one coordinate system, it is zero in every coordinate system. And it is simply not true with that T_{mn} – not because the vector may change from point to point, but because the orientation of the coordinates changes.

So we need a better definition of the derivative of a vector than just differentiating its components. We need something which if it is zero in one frame is zero in every frame.

Here is how we will define the derivative of a vector. As a preliminary, notice that to define the derivative at a point P we only need to look at points in the vicinity of P.

The first thing we do is to construct a set of Gaussian normal coordinates at point P. Remember : Gaussian normal coordinates are as straight as possible near P. They are well defined over the whole variety, and they make up an approximately Euclidean system of coordinates in the vicinity of P. So we re-express all the vectors of the vector field in the new coordinates X that are locally flat and Euclidean.

To follow the procedure geometrically, let's look again at two vectors, one at P and the other at Q nearby, and for clarity make the second also slightly different from the first.

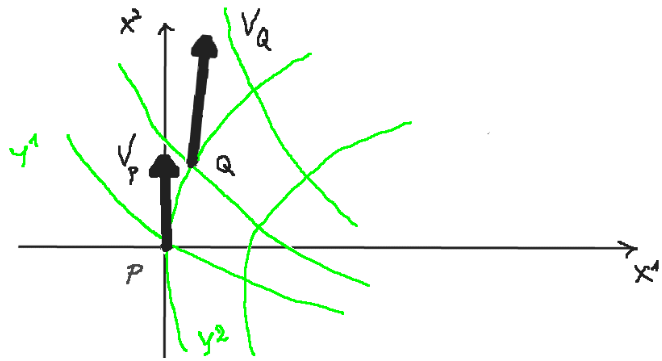

Figure 7 : Two vectors nearby.

Then pretend that the Gaussian normal coordinates are really nice flat coordinates over the whole vicinity of P. Visually translate the vector V_Q so that its origin be the same as V_P's, and look at the difference between V_Q and V_P.

The components, in Gaussian normal coordinates, of the difference between V_Q and V_P are the kind of elements that we will use to define the derivatives of the vector field at P. For instance if we were interested in the derivatives along the PQ direction, it would approximately be $\overrightarrow{V_Q - V_P}$ divided by the small distance between P and Q. But we shall be interested in the derivatives of V along the X axes.

Now the derivatives $\partial V_m / \partial X^r$, in the Gaussian normal coordinates, *define* the derivatives of V at P.

Finally, if we want to work with our initial Y coordinates, we take the double-indexed collection of partial derivatives produced by the differentiation in Gaussian normal coordinates at P. We treat it as a tensor, that is, we consider this collection as the components of a tensor in the X coordinates. And we transform it back into the Y coordinates using the tensor equations linking X and Y.

This necessarily produces a tensor, since when transformed from any Y coordinates into the X coordinates, it gives the same thing.

Now when we look at the derivatives of a vector in the general coordinates Y, we will have the addition of two terms : one term because the vector may be changing and the other term because the coordinates may be changing. As we saw, the coordinates may shift, may be rotating out from under you, even if the vector doesn't change (see figures 5 and 6).

But before we look into these two terms, let's repeat the prescription as a methodical procedure.

You have a vector field V on a variety, equipped with any coordinate system Y. You want to calculate the derivative of V at P. Then follow these steps :

1. Change coordinates to use Gaussian normal coordinates at P, let's call them X (notice that they are valid over the whole surface, and approximately flat at P).
2. Differentiate V at P in the usual way, using the X coordinates.
3. Consider the collection of partial derivatives you got as the components of a tensor of rank 2 in the X coordinates.
4. Switch back to your original coordinate system Y, and re-express in that original system the tensor you got, using the tensor equations linking X and Y.

Let's look at what we get and then comment on each term. We find that, with this new definition, the derivatives at P are the old ones corrected by the addition of another term :

$$D_r V_m = \partial_r V_m - \Gamma^t_{rm} \, V_t \tag{13}$$

Here is how to read equation (13) :

1. The notation $D_r V_m$ is by definition the partial derivative of V_m with respect to the r-th direction in X obtained from the above procedure, that is, in the Gaussian normal coordinates X, and then re-expressed in the Y coordinates.

2. $\partial_r V_m$ is the ordinary partial derivative of V_m with respect to the r-th direction in Y calculated directly in the Y coordinates. ∂_r is a short hand for $\frac{\partial}{\partial Y^r}$.
3. $-\Gamma^t_{rm}\ V_t$ is the additional term due to the fact that the coordinates Y themselves evolve in the vicinity of P. The minus sign is a pure convention. This whole second term on the right hand side of equation (13) must clearly be proportional to V_t. If you double the size of V_t it must be twice as big. And the coefficient Γ^t_{rm} in front of V_t is a new mathematical object appearing in the differentiation procedure. We shall talk about it.

The term $\Gamma^t_{rm}\ V_t$ does not have a derivative related to the vector because if doesn't come from the fact that the vector is changing. It comes from the fact that the coordinates are changing in the vicinity of P.

The right hand side of equation (13) is what you will get if you take a vector, differentiate it in Gaussian normal coordinates and then transform the double-indexed collection of derivatives to other coordinates as a tensor. In any other coordinate system what you will get is the usual derivative in that coordinate system *minus an object* times the components of V themselves. As usual $\Gamma^t_{rm}\ V_t$ means of course the sum over t.

Equation (13) holds in any arbitrary coordinate system. Of course the V_m's or V_t's and the Γ^t_{rm}'s depend on the coordinate system. Notice that there is not only one Γ^t_{rm} : they form a three-indexed collection.

$D_r V_m$ is called the *covariant derivative* of V_m [6]. This is not because the index m is downstairs. The terminology has another origin. If we had differentiated V^m, we would have obtained a formula analogous to equation (13). And it would also be called the covariant derivative of V^m.

And, by its very construction, $D_r V_m$ *is a tensor.*

Christoffel symbols

The coefficients Γ^t_{rm} have two names : *connection coefficients* and *Christoffel symbols.*

The name *connection coefficients* comes from the fact that they connect neighboring points and tell us how to calculate the rate of change of a vector field from one point to another, even though the coordinate system may be changing.

They are also named *Christoffel symbols* after Elwin Bruno Christoffel [7]. They have been known on occasion as the "Christ awful" symbols because they seem complicated. With some practice, however, the reader will discover that they are not that complicated. They are just an extra linear term. But I grant you that they are complicated and

6. In other contexts it goes under the names of *material derivative* or *convective derivative* or *Lagrangian derivative*, etc.

7. Elwin Bruno Christoffel (1829-1900), German mathematician at the University of Strasbourg – when it was in the German empire between 1871 and 1918 – who did fundamental work in differential geometry.

unlikeable enough.

Let's investigate what follows from the definition of the covariant derivative and the Christoffel symbols. We are not going to prove every single fact we state, because there are just too many little pieces. But they are easy to check.

It follows from the definition of the covariant differentiation – namely, to differentiate a vector V at a point P, go to a set of Gaussian normal coordinates at P, differentiate the vector in the ordinary manner, treat the object you obtain as a tensor with two indices, change coordinates, etc. – that the Christoffel symbols have a symmetry :

$$\Gamma^t_{rm} = \Gamma^t_{mr} \tag{14}$$

There are generalized Riemannian geometries, also called geometries with torsion, in which this symmetry is not true. But those geometries are not widely in use in ordinary gravitational theory. The geometry of general relativity is the Minkowski-Einstein geometry which is an extension of Riemannian geometry with a non-positive definite metric. But it doesn't involve torsion. So the Christoffel symbols we will use will be symmetric in the sense of equation (14).

A remark to build our physical intuition : doing the derivative in Gaussian normal coordinates which are almost flat, or as flat as can be, and then treating what we obtain as an object in its own right, is very similar to what we do in gravitational theory when we evaluate something in a freely falling frame.

For example in a freely falling frame we calculated how light moved across an elevator and then we transformed it to the frame of reference in which the elevator was accelerating.

That is closely related to the operations we have been doing in this chapter : we calculate something because we know how to do it in coordinates which are as flat as possible. That would be a freely falling frame in general relativity. And then we transform it in any coordinate we like, accelerated coordinates or anything we like, and we translate the statement from one coordinate system to another.

In the construction of the covariant derivative, the calculation of the variation of a vector from point to point is done first in Gaussian normal coordinates, and then it is transformed in any coordinate system. Equation (13), reproduced below,

$$D_r V_m = \partial_r V_m - \Gamma^t_{rm}\ V_t \tag{15}$$

is the form that you get for the corresponding collection of components. It is a tensor. However, $\partial_r V_m$ is not a tensor. Therefore $\Gamma^t_{rm}\ V_t$ cannot be a tensor. And Γ^t_{rm} cannot be a tensor either.

We will see that the Γ^t_{rm}'s are build-up out of the derivatives of the metric $\partial_r g_{mn}$. In fact in a coordinate system in which the derivatives of the metric are 0, the Christoffel symbols are 0. But a tensor, if it is zero in one coordinate system, it is 0 in every coordinate system. So that is another way to see that they can't be tensors.

Let's look now at the covariant derivative of higher rank tensors, because we will need this for curvature. Suppose we have a tensor with more than one index, say

$$T_{mn}$$

and we want to differentiate it covariantly along the r-th axis. We denote the resulting tensor

$$D_r T_{mn}$$

Its expression is the analog of equation (15), except that for every index in the tensor T_{mn} there will be a term like $\Gamma^t_{rm}\ V_t$. Let's see you how it works :

We start by working only on the m index, letting n be passive, and writing the equivalent of (15)

$$D_r T_{mn} = \partial_r T_{mn} - \Gamma^t_{rm} T_{tn} - ...$$

But we are not finished. We have to do exactly the same with the n index, letting this time m be passive.

$$D_r T_{mn} = \partial_r T_{mn} - \Gamma^t_{rm} T_{tn} - \Gamma^t_{rn} T_{mt} \tag{16}$$

That is the form of the covariant derivative at point P of the tensor T_{mn}. The rule is the same : we switch to Gaussian normal coordinates at P, we do the ordinary differentiation of the tensor with respect to each direction X^r. This adds one more index to the collection of components that formed T_{mn} [8]. And we re-express the new tensor in the original coordinate system with the usual tensor equations (equations (15) and (16) of chapter 2 and their generalizations).

This allows us to to differentiate any tensor. At the moment we are only dealing with tensors with covariant indices. We will come in a moment to tensors with contravariant indices.

8. Notice that in order not to clutter equations we no longer use prime signs for one system and un-prime for the other. The last time we did it in this chapter was in our comments following figure 6.

The reader may wonder : what is all this intricate business of covariant differentiation of tensors for ? Answer : it is for comparing things at different points. We want to be able to talk about rates of variation of things along coordinate lines, with objects which have an existence irrespective of the system of coordinates we work with.

Remember that a vector in ordinary 3D has an existence irrespective of the basis we are using. For certain work and calculations with it – not all of them – we need a representation of the vector in a basis. The collection of components to represent it and work with it is different from one basis to another, but the vector we are talking about is the same [9].

Where are we going to use covariant derivatives ? Answer : in field equations. Field equations are going to be differential equations which represent how a field changes from one place to another. But we want them to be the same equations in every reference frame. We don't want to write down equations which are special to some peculiar frame. We want them to be valid in general. That is, if they are true one frame they will be true in all frames. That means they have to be tensors equations. So we have to know how to differentiate tensors to get other tensors.

Another point worth stressing : the Christoffel coefficients will be present in equation (16) even in a flat space – like a plane, or this page, or the ordinary 3D Euclidean space – *if you chose funny coordinates* (see figure 3 of chapter 2). That is an important point : terms like $\Gamma^t_{rm} T_{tn}$ are there

9. It is even like the difference between things and their notations. When we talk about 12 in the decimal system or 1100 in the binary system, we are actually talking about the same thing : the number twelve – which some other people call douze...

even in flat space if you are using funny coordinates. In fact if you choose any coordinates in which the derivatives of the g_{mn}'s are not zero, that is, in which the coordinates vary from point to point sinuously (viewed from an embedding space for instance), $\Gamma^t_{rm}T_{tn}$ will be there.

The presence of terms like $\Gamma^t_{rm}T_{tn}$ in the covariant derivative of a tensor is not a characteristic feature of curved spaces, it is a feature of curved coordinates.

To begin to use our new tool, let's apply equation (16) to the metric tensor itself. There is something special, however, about the metric tensor : in Gaussian normal coordinates, its derivatives are all zero. That means that the covariant derivative of the metric tensor is zero. That simple observation is what is going to allow us to compute the Christoffel symbols. Let's write equation (16) with the metric tensor. We are in a coordinate system that is general, curvilinear or not, flat of not, whatever :

$$D_r g_{mn} = \partial_r g_{mn} - \Gamma^t_{rm} g_{tn} - \Gamma^t_{rn} g_{mt}$$

But we know that this is zero. Why ? Because the ordinary derivative of the metric tensor in Gaussian coordinates is zero. So, in any coordinate system, we have

$$\partial_r g_{mn} - \Gamma^t_{rm} g_{tn} - \Gamma^t_{rn} g_{mt} = 0 \tag{17a}$$

Now let's write the same equation, except permuting the indices. It is a little trick to get as much information about the Christoffel symbols as we can, and eventually, via some nice eliminations, to be able to isolate one Christoffel symbol and express it in terms of the ordinary partial derivatives of g with respect to the axes in any coordinate system.

So equation (17a) becomes

$$\partial_m g_{rn} - \Gamma^t_{mr} g_{tn} - \Gamma^t_{mn} g_{rt} = 0$$

But the middle term, by symmetry, can be rewritten interchanging m and r

$$\partial_m g_{rn} - \Gamma^t_{rm} g_{tn} - \Gamma^t_{mn} g_{rt} = 0 \tag{17b}$$

And for exactly the same reasons we can write

$$\partial_n g_{rm} - \Gamma^t_{rn} g_{tm} - \Gamma^t_{mn} g_{rt} = 0 \tag{17c}$$

Let's write the three equations of interest next to each other, to look at them more conveniently :

$$\begin{aligned} \partial_r g_{mn} - \Gamma^t_{rm} g_{tn} - \Gamma^t_{rn} g_{mt} &= 0 && \text{(a)} \\ \partial_m g_{rn} - \Gamma^t_{rm} g_{tn} - \Gamma^t_{mn} g_{rt} &= 0 && \text{(b)} \\ \partial_n g_{rm} - \Gamma^t_{rn} g_{tm} - \Gamma^t_{mn} g_{rt} &= 0 && \text{(c)} \end{aligned}$$

How can we add them, or subtract them, or do something clever, that will isolate only one of the terms with a gamma ?

Let's add equation (b) to (c) and subtract (a). Of course we will get $\partial_n g_{rm} + \partial_m g_{rn} - \partial_r g_{mn}$ plus some other terms. But the middle term of (a), $\Gamma^t_{rm} g_{tn}$, will disappear, and so will the last term of (a), $\Gamma^t_{rn} g_{mt}$. And we will be left with twice the same last term with a gamma, $\Gamma^t_{mn} g_{rt}$. So we are in luck : (b) + (c) − (a) yields

$$\partial_n g_{rm} + \partial_m g_{rn} - \partial_r g_{mn} = 2\Gamma^t_{mn} g_{rt} \tag{18}$$

We are still not done. We would like to get Γ^t_{mn} by itself. Our goal, indeed, is to find out what the Christoffel symbols are in terms of derivatives of the metric. We are almost there.

Fist of all, equation (18) shows that if all the derivatives of the metric are zero, the Christoffel symbols must be 0.

But how are we going to get rid of the g_{rt} on the right hand side of equation (18) ? The answer comes from recalling that g_{rt} has an inverse. We saw that in the form of matrix equations, as well as in the form of tensor equations, see equation (33) of chapter 2. We multiply both sides of equation (18) by the inverse tensor, and move also the factor 2, and we get

$$\Gamma^t_{mn} = \frac{1}{2}\ g^{rt}\ \left[\ \partial_n g_{rm} + \partial_m g_{rn} - \partial_r g_{mn}\ \right] \tag{19}$$

This is the expression of the Christoffel symbols in terms of the ordinary derivatives of the metric tensor.

It looks pretty simple. The indices m and n are symmetric. You can interchange them, the Christoffel symbol won't change. There is one negative term and two positive terms. It is not very complicated.

The problem is that there is a boatload of them. When you think about a four dimensional space and let all the coefficients range from 1 to 4, there is just a lot of Christoffel symbols.

That is what makes doing calculations in general relativity a very tedious business. Intrinsically there is nothing hard about it. But in fact doing a calculation in a general relativity context usually fills page after page of nothing more complicated than just computing these derivatives and assembling them together.

Equation (19) holds for any coordinate system and any metric tensor. Notice that all our calculations are *at one point* P. Whatever coordinate system our variety is equipped with, we position ourselves at a point on it, consider the metric tensor g_{mn} there, calculate the gammas there with equation (19).

The use of Gaussian normal coordinates at P was just for intermediate reasoning, calculation and proof purposes. We are now back in the initial coordinate system of our space.

The g_{mn}, g^{mn} and Γ^t_{mn} depend on P. But equation (19) is general. At every point, it expresses the connection coefficients – the other name of the Christoffel symbols – in terms of the derivatives of g. These connection coefficients enable us to figure out how any vector or tensor varies, in our space, when we move a little bit along the coordinate lines.

The problem with the Christoffel symbols is that they are not tensors. They can be zero in one frame of reference, and not zero in another. For example, in a set of Gaussian normal coordinates at point P, all of the Γ^t_{mn} are equal to zero. This can be seen in many ways. Since the metric tensor in that case is constant (even equal to the Kronecker delta tensor, but that is not necessary), equation (19) tells us that $\Gamma^t_{mn} = 0$. But in some other coordinate systems the Christoffel symbols are not.

Remember that even in an intrinsically flat space, we can have coordinates such that the metric tensor is not constant. Then the Chritoffel symbols won't be zero. As said, they are related to the coordinate system not to the intrinsic geometry of the space.

A sphere is intrinsically non flat. In the polar coordinates θ and ϕ (see chapter 1, figure 14), the components of g are not constant, therefore the Christoffel symbols are not zero in that system of coordinates. Even on a sphere, however, at any given point we can build a set of Gaussian normal coordinates (like maps do – you just tinker with the longitude), and then the Christoffel symbols at that point will be zero.

Exercise 1 : Explain why the space can be flat and nevertheless the Christoffel symbols not zero.

Exercise 2 : Explain why the covariant derivative of the metric tensor is always zero.

Exercise 2 : On the Earth, with the polar coordinates θ for latitude, and ϕ for longitude, find

1. **the metric tensor g_{mn}**
2. **its inverse g^{mn}**
3. **the Christoffel symbols at point $(\theta,\ \phi)$.**

All this is conceptually tricky when we meet it for the first time. But at the end of the day the rule is simple : calculate the Christoffel symbols and, in many contexts, replace ordinary derivatives with covariant derivatives.

You could write your equations in Gaussian normal coordinates. Then they would just involve ordinary derivatives and we would not have to wade through a river of

Christoffel symbols. But if you want the same equations in general coordinates then replace the ordinary derivatives by covariant derivatives.

That is the procedure. It will require the reader to think about it. He or she will have to sit down, to follow carefully the reasonings, to do exercises. Then what we are doing will become clear.

Curvature tensor

What is curvature ? It is easiest to start with two-dimensional curvature. Intuitively it is easy to understand : it is a characteristic of something that is round and cannot be flattened out. But we are going to give it some more mathematical definition. How do we probe for curvature ?

Let's begin by drawing a space which is curved. The sphere is obviously curved. I don't want, however, to deal with a sphere. I want to look at a cone.

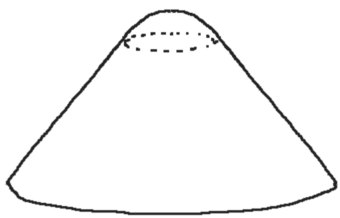

Figure 8 : Cone with a rounded summit.

But it is going to be a cone with a round summit, figure 8. Think of the top of a mountain the sides of which are nice and flat like those of a volcano, and the top is round.

So if you are away from the top of the mountain, below the dotted line, around you the surface is flat. It may not look flat because, like the furled page in figure 1 of chapter 2, we represented the mountain embedded in 3D Euclidean space. But the surface is what mathematicians call *developable* : any section with no hole in it, cut from the side of the mountain, can be flattened onto a plane without distorsion.

The rounded cone only differs from a flat space in this vicinity of the summit. To see that just take the same space below the dotted line but continue it so that it really does form a genuine cone.

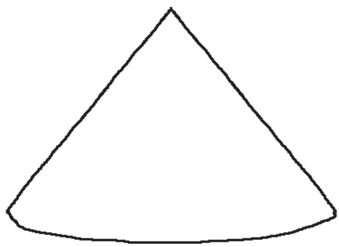

Figure 9 : Genuine cone.

Then slice it along a generatrix, that is, a straight line on the cone going to the top. And open it up. You can lay out flat on a plane the shape that you get. It is a disk with a missing piece, see figure 10.

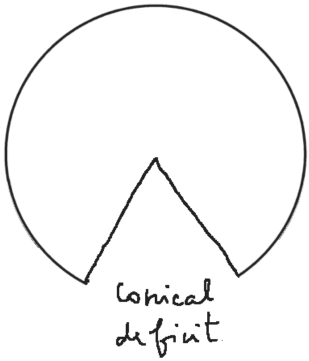

Figure 10 : Cone opened up and laid flat
(smaller scale than figure 9).

The missing piece is called the deficit angle, or the *conical deficit*. We can see that the bigger the conical deficit is, the pointier the cone will be.

Now, *on the flat surface* of figure 10, let's consider a collection of identical vectors arranged around the shape as shown in figure 11.

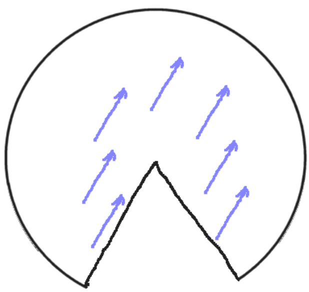

Figure 11 : Identical vectors.

On the flat surface, all the vectors point in the same direction. But when we fold again the shape to form the cone, we see that the vectors no longer point in the same direction. Think of them as very small so that they don't have to be bent. The first one on the left is along a generatrix, but the last one on the right points in another direction.

So the tip of the cone is such that if you carry a vector around (in the same direction on the flattened version) when you get around to the other side it has been rotated. The rotation is due to the conical deficit.

Exactly the same is true on the rounded cone of figure 8 : if we take a vector on the flat side, below the dotted line, and we carry it around the mountain in such a way that, when it is opened up and laid on a plane, it is always pointing in the same direction, by the time we get back to the other side it will be pointing in some other direction. That is because the summit of the shape is curved. That is the effect of curvature.

A region of curvature has the property that if you take a vector pointing in the same direction as much as you can do, that is locally parallel to itself, and take it around the region of curvature, then the vector will rotate.

The vector will rotate when it goes around any region of curvature even though you took pains to make sure that at every point you were moving it parallel to itself. In other words you took pains to make sure that *the covariant derivative of the vector was always zero*. But nevertheless, when you came around to the other side, it is shifted.

There is another way to say this, which is equivalent and actually more useful.

Consider a curved space with some curvature at point P, figure 12.

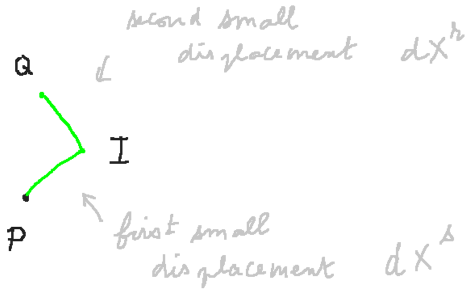

Figure 12 : Displacements to differentiate a vector field along two axes.

Take a vector field and differentiate it along one axis (first displacement in figure 12). Then differentiate it along the second axis (second displacement in figure 12). That is, you consider the vector field at P. Then you move a bit along one axis, and consider the new value of the vector field at I. Then you move another bit along the second axis, and consider the value of the vector field at Q.

Over each displacement, the vector will change. How will it change ? It will change typically by differentiating the vector along the two axes in sequence. We first differentiate the vector along one axis and then differentiate it along the second axis. And this will produce a small change in the vector due to the two derivatives.

The total change in the vector consists of two changes. And that total change is proportional to a second derivative [10]. That is true in any coordinates : if, to compare the vectors at Q and at P, you compared the vector at I with the vector at P, and then compared the vector at Q with the vector at I, what you would be calculating is the second partial derivative of the vector with respect to the two directions.

In figure 12, if the first displacement is along the direction X^s and the second displacement along the direction X^r, then the variation of the m-th component of the vector V – let's say it has covariant indices – would be

$$D_r D_s V_m \tag{20}$$

This expression is calculated covariantly. And in Gaussian normal coordinates, expression (20) would just contain ordinary derivatives.

We could have also gone in the other direction, figure 13. That is to say we could have gone first in the r direction and then in the s direction and calculated the way the vector changes from P to J and then from J to Q. The variation of V would then be

$$D_s D_r V_m \tag{21}$$

10. To get a feel for this, think of a function f of two independent variables a and b. View it in 3D. Suppose f is nice and smooth everywhere, and in particular, at the point $(0,0)$, is tangent to the plane formed by the a and b axes. Suppose we are interested in $f(a,b) - f(0,0)$. If a and b are small, this variation is approximately equal to $ab\frac{\partial^2 f}{\partial a \partial b}(0,0)$.

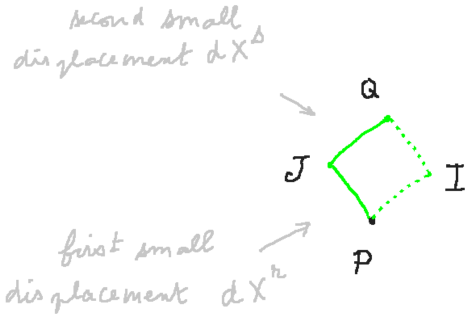

Figure 13 : Displacements in the other order.

Ordinarily, and in flat space in general, expression (20) and expression (21) are equal to each other. That is

$$D_r D_s V_m = D_s D_r V_m \tag{22}$$

This is just a version of the fact, in calculus, that the partial derivatives of a function of several variables can be taken in the order you like (see Interlude 3 of Volume 1 in the collection *The Theoretical Minimum*).

But equation (22) is not true in curved space. In curved space the difference

$$D_r D_s V_m - D_s D_r V_m$$

can be thought of as taking the vector around the closed loop

$$P \to I \to Q \to J \to P$$

Let's go back to our cone, either the genuine cone (figure 9), or the cone with a rounded top but looking at the part below the dotted line (figure 8). Consider a vector field which when the cone is opened and laid flat is constant. Fold the flat shape to form the cone. We discovered that if we follow the vector field on a closed loop around the top, we don't get back to the same vector we started with. This is due to the following fact, that is important enough to note with italics.

In curved space covariant derivatives are not interchangeable. In flat space they are interchangeable.

That is the way we will test whether the space is flat or not.

We will test whether differentiating tensors, and in particular vectors, in opposite order gives the same result.

- If the answer is yes everywhere in the space for any vector, then the space is flat.
- If we discover that there are places in the space where the order of differentiation gives different answers, then we know the space has some kind of defect in it (like the point of the genuine cone) or has curvature (like the summit of the rounded cone).

So all we have to do is compute the second covariant derivatives of a vector in opposite order and compare them. In principle it is not complicated. In practice it will be a little complicated, but will remain manageable. We have all the tools to do it. Now it is a mechanical operation, consisting of pure plug-ins. We sketch the steps below, and then give the answer.

We start with a vector expressed with covariant components

$$V_n$$

We compute its covariant derivative in the r direction

$$D_r V_n$$

And then we differentiate this, still covariantly, in the s direction

$$D_s D_r V_n \tag{23}$$

Later on, we will interchange s and r and subtract.

Let's replace the first covariant derivative of V_n, with respect to r, by its expression given in equation (13). We get

$$D_s D_r V_n = D_s \left[\partial_r V_n - \Gamma^t_{rn} V_t \right]$$

But notice that $\left[\partial_r V_n - \Gamma^t_{rn} V_t \right]$ is a tensor. So we know how to differentiate it : use equation (16). Continue to crank mechanically the calculations.

In the end, the difference between the two second order covariant derivatives yields a tensor, denoted $\mathcal{R}_{srn}^{\ \ \ t}$, multiplied by V_t

$$D_s D_r V_n - D_r D_s V_n = \mathcal{R}_{srn}^{\ \ \ t} V_t \tag{24}$$

And here is $\mathcal{R}$

$$\mathcal{R}_{srn}^{\ \ \ t} = \partial_r \Gamma^t_{sn} - \partial_s \Gamma^t_{rn} + \Gamma^p_{sn} \Gamma^t_{pr} - \Gamma^p_{rn} \Gamma^t_{ps} \tag{25}$$

There are two terms involving derivatives of Christoffel symbols, and two terms which are sums over p of products of Christoffel symbols.

$\mathcal{R}_{srn}^{\ \ \ t}$ is the *curvature tensor*, also called *Riemann curvature tensor* or *Riemann–Christoffel tensor*.

It has a complicated expression. And it is even more complicated when you remember that the Christoffel symbols are given by equation (19), which we reproduce below

$$\Gamma^t_{mn} = \frac{1}{2}\, g^{rt} \;\left[\, \partial_n g_{rm} + \partial_m g_{rn} - \partial_r g_{mn} \,\right]$$

Let's see what are the elements in the curvature tensor given by equation (25).

The Christoffel symbols involve derivatives of g. So differentiating again produces second derivatives of g. Remember : the second derivatives of g are the things that we cannot generally set equal to zero.

For the first derivatives of g, we saw that we can find a frame of reference where they are equal to zero. But for the second derivatives of g we can't. So by the time we are finished calculating the curvature tensor, the second derivatives of g have come into it.

The second derivatives are testing and probing out the geometry of the surface a little more thoroughly than just the first derivatives. In a similar way, in the theory of functions, when at a point x you know $f(x)$ and $f'(x)$ and $f"(x)$, you are better off than with just the value of f and of its first derivative.

So the curvature tensor has second derivatives of the metric g, and it has squares or quadratic things involving first derivatives of g. It is a complicated thing. If we were to actually write it in terms of the metric, or we were to try to calculate it for a given metric, it could rapidly fill up pages.

But conceptually what it is doing is simply calculating the difference in a vector if you transport it around the loop in figure 13, keeping it parallel to itself, as much as you can locally at every point, until you have come all the way around. It calculates the little change in a vector in parallel transport going around a loop.

The curvature tensor has a complicated formula, but we can calculate it. We can put the metric tensor into a computer and ask the computer : "Is the curvature tensor 0 ?" It is even better if you have a software that can do algebra. If you have the metric in some algebraic form, you can do all the operations of equations (19) and (25) and then test out whether the curvature tensor is zero everywhere. If the curvature tensor is zero everywhere, that is all its components are zero everywhere, then the space is flat.

We shall study the curvature tensor a little more. As we said, it is a complicated thing. Its main use is to tell us whether the space is flat. And, if not, how unflat it is.

It is closely related to a quantity in gravitational physics. Can you guess which one ? A local quantity which tells you that the space is not flat. It must be something telling you whether there is really a gravitational field present or not.

Answer : tidal forces. It is exactly related to tidal forces, those things which in a gravitational field squeeze bodies one way and stretch them another way. Tidal forces are represented by the curvature tensor.

Here is another way to get a feel for what the curvature tensor is. Imagine a surface which is flat away from a point in the center where it has a bulge. It doesn't have to be a rounded cone. It can simply be a plane with a bulge, as in figure 14.

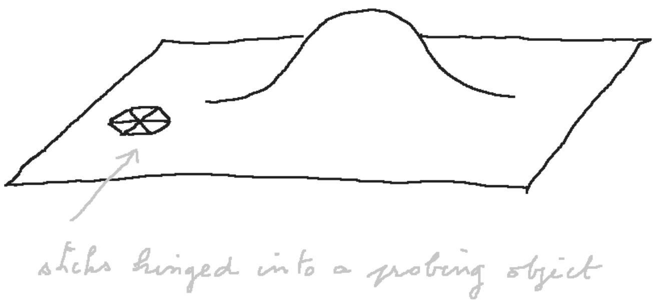

Figure 14 : Probing the curvature with
a little structure of Tinkertoy sticks.

And you have a small structure of Tinkertoy sticks, all hinged at their extremities, so that their directions can move freely from each other, while remaining attached. At first, the probing structure lays flat, without stress or distorsion, in a flat part of the surface, because the probe is itself flat.

And then you start moving the probe. While you move it in the flat region, nothing happens to it. It remains perfectly happy. It doesn't get stretched, it doesn't get distorted or deformed. This would have also been the case on the side of the rounded cone away from the summmit, by the way.

But now what happens when you try to move the probe into the curved region ? When you are trying to move it into the curved region, it simply can't follow the curvature without having to stretch or compress some of its lengths. It has to follow the metric properties of the curved space.

In particular if you go around the probe, what you are doing somehow is sampling the double covariant derivatives of equation (20) or (21). You are going to find out that various angles between sticks change from their value in flat space. The lengths of the sticks shift too, they get stressed, they get deformed.

The measure of how much the probe gets stressed locally is given by the curvature tensor. The curvature is an important property because, if you are in region where there is curvature, you can feel it, either with tidal forces in a gravitational field, or with the probe in the experiment of figure 14.

Uniform gravitational fields don't have curvature. That is why in free-fall, in a perfectly uniform gravitational field, you simply feel nothing. Indeed they don't cause tidal forces. Of course perfectly uniform gravitational fields don't really exist in nature. You can simulate one with acceleration, but you cannot see one in nature. They exist only approximately on the surface of big massive objects, if you limit yourself to a small solid angle. This leads us to a last remark.

Tidal forces, or curvature on a surface, have a bigger effect on bigger objects. The 2000-mile man in free-fall toward the Earth will feel tidal forces more strongly than a free-falling bacteria. Similarly, in figure 14, if the probe is small compared to the bulge, it won't be much deformed when it goes over it. Whereas if it were a bigger Tinkertoy structure, made for instance of many more hexagones, covering a larger region of the plane, like floor tiles, but still hinged so that any two connected sticks can change their directions from each other, then the probe would feel the curvature more strongly.

At the end of this third chapter we have reached the curvature tensor. It is complicated. Its expression is given by equation (25). I wish we could do without the sea of symbols and indices.

But we can compute it. Often you will be presented with the metric tensor in some analytic form. There will be a formula for it. With the formula you can do differentiations. Everything consists in analytic functions that you can calculate. So we have finally reached our initial aim.

Remember that our aim was to find a method to determine whether a space is flat. By definition, it is flat if there exist a set of coordinates in which the metric tensor is everywhere equal to the Kronecker delta tensor. But trying out every possible set of coordinates, and checking them at every point, was not a practical solution. So we found the curvature tensor. If it is zero everywhere, then we can find a set of coordinates such the metric tensor is everywhere equal to the Kronecker delta tensor. You just position yourself at any fixed point and start to build Euclidean coordinates, like we did when we built Gaussian normal coordinates. Now no curvature will limit us to a small vicinity.

In summary, *the curvature tensor has a complicated form but it is practical tool.*

We are finished with our mathematical study of Riemannian geometry, metrics, tensors, curvature, etc. The interested reader who wants to go further into the mathematical aspects of these topics is invited to take up any good manual in differential geometry oriented toward applications. As far as we are concerned, our new tools will now be put to use.

In the next chapter, we will enter into gravity land. We will see what has to change to go from Riemann geometry to Einstein geometry. Then we will study a particular example : the Schwarzschild geometry. It is the geometry of a black hole, a star, or any gravitating mass.