Lesson 5 :
Metric for a gravitational field

Notes from Prof. Susskind video lectures publicly available on YouTube

Space-like, time-like, and light-like intervals, and light cones

Let's begin with time-like, space-like, and light-like intervals. For that we go back to special relativity to spell out what that means.

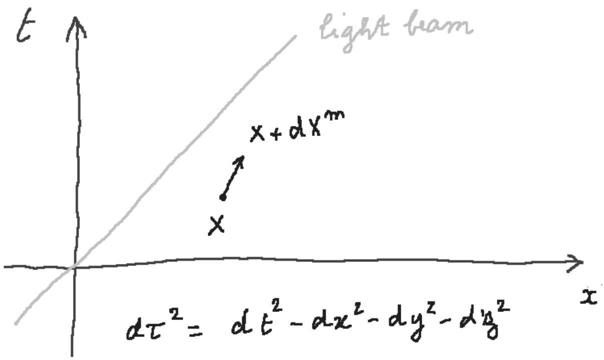

Figure 1 : Space-time and Minkowski metric.

We have discussed the metric many times. We call it proper time

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

Sometimes we want to put in explicitely the speed of light. It then becomes

$$d\tau^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} \tag{2}$$

The reason to introduce the speed of light is to keep track of what is small and what is big under certain circumstances. For example if we want to go to the non-relativistic limit, that is, the limit where everything is moving slowly, it is good to put back c because it reminds us that it is much bigger than any other velocities in the problem. And it makes it easy to see which terms can be neglected and which terms cannot. So as before we will sometimes put it in and sometimes take it out depending on circumstances.

Let's look at the sign of $d\tau^2$. Of course, when we look at real numbers, their square is always positive. But $d\tau^2$ is not defined as the square of a real number, it is defined by equation (2). And it can be positive or negative, depending on whether $dx^2 + dy^2 + dz^2$ is smaller than dt^2 or bigger than dt^2. Of course when we say $dx^2 + dy^2 + dz^2$, we mean $(dx^2 + dy^2 + dz^2)/c^2$.

If $d\tau^2 > 0$, then the little element dX^m in figure 1 is said to be *time-like*. It has more time than it has space so to speak. Its vertical part is bigger that its horizontal part. Its slope in figure 1 is greater than 45°.

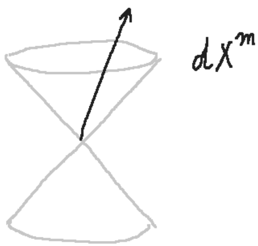

Figure 2 : Time-like interval.

It can be described in terms of a light cone, figure 2. If we represent two spatial coordinates x and y, in addition to the time coordinate t, and a light cone whose center is at

X, then it means the little vector dX^m lies in the interior of the cone.

It could also lie in the backward direction in the same picture, pointing to the past. In either case, it is called a time-like interval.

Space-like is exactly the opposite of time-like. It corresponds to $d\tau^2 < 0$, or equivalently $dx^2 + dy^2 + dz^2$ greater than dt^2. In that case we usually define another quantity dS^2 which is just the negative of $d\tau^2$. By definition it is

$$dS^2 = dx^2 + dy^2 + dz^2 - dt^2 \tag{3}$$

It is the same object as $d\tau^2$ except for the minus sign. Space-like vectors are those for which $dS^2 > 0$. And if we represent the cone at X as in figure 2, we get figure 3.

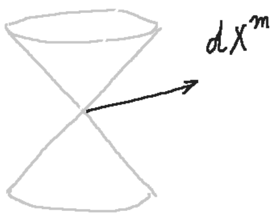

Figure 3 : Space-like interval.

Finally there are *light-like* vectors. They are those for which both $d\tau^2 = 0$ and therefore $dS^2 = 0$. Their slope, in the standard diagram, is at 45°. They are trajectories of light-rays, and they lie on the surface of the cone of figure 3.

Those are the three kinds of vectors we can have.

Just for a moment consider what it would mean, if there were two positive signs and two negative signs instead of one positive sign and three negative signs in the definition of the metric in equation (1). Somehow this would correspond to two time directions. It doesn't mean anything in physics. There are never two time directions. There are always one time and three space. Can you imagine a world with two times ? I can't. Frankly I can't. I cannot imagine what it would mean to have two different time directions. And we will simply take the view that that is not an option. There is always one time-like direction in the metric of equation (1) and three space-like.

But that doesn't mean that there is a unique direction which is time-like. There are many time-like directions pointing within the light cone of figure 2.

The invariant property, corresponding to the fact that at any point there is one time and three space directions, concerns the metric tensor. We can write the metric tensor in the following expression

$$d\tau^2 = -g_{\mu\nu}\ dX^\mu\ dX^\nu \tag{4}$$

The minus sign in front of $g_{\mu\nu}$ is a convention, so that dS^2, the square of the infinitesimal *proper distance*, is given by the same expression but without the minus sign.

Let's recall this important point about the *proper time* $d\tau$, defined by equation (4) : it is the time recorded by a clock accompanying the particle along its trajectory. In other words, it has a physical practical meaning which is often useful to remember. Of course, as we know, for particles going slowly – and by slowly we mean up to thousands of miles per second – the proper time is essentially the same as the standard time of the stationary observer in the stationary frame of figure 1. This is easily seen from equation (2), because c is very big compared to ordinary velocity, or to the spatial components of the 4-velocity. Go to the volume 3 on special relativity, in the collection *The Theoretical Minimum*, if you need a brush up on these ideas.

Similarly, *proper distance*, dS, along the trajectory of a particle is distance measured by a meter stick carried along by the particle.

In summary, proper distance, $\sqrt{dS^2}$, really is a distance. And proper time, $\sqrt{d\tau^2}$, really is a time. Let's keep that in mind.

Equation (3) is the definition of the metric with the coordinates $(t,\ x,\ y,\ z)$. It can always be written in terms of a matrix. In this case it is the matrix η written below

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Remember that it is the analog in the Minkowski space of the Kronecker delta, that is simply the unit matrix, in Euclidean space.

The matrix η has obviously three positive eigenvalues and one negative eigenvalue. And that is the invariant story.

The story is not that there is only one time direction, but that *there is always only one negative eigenvalue in the metric*. In special relativity and in general relativity, that will still be the case whatever the metric is and whatever the coordinate system is. And since the metric, in general, depends on the point in space-time where we look at it, this invariance statement will be true at any point.

A metric which would have two negative eigenvalues or three negative eigenvalues, would have more than one time. And we just don't even think about that. That is something that physics does not seem to have made use of, several time axes.

The concepts of time-like, space-like and light-like displacements are not restricted to special relativity. They apply generally whatever the metric, and whatever the point which we consider. In the preceding discussion, we were in the flat space of special relativity, but the concepts apply in general relativity where the space is intrinsically not flat.

Now we shall consider a metric more general than $\eta_{\mu\nu}$. We consider $g_{\mu\nu}(X)$. At every point, that is, at every X, there is a matrix. And the matrix must have one negative eigenvalue and three positive eigenvalues.

In other words, wherever you are – standing there – you should experience a world with one time direction and three space directions, or more exactly one negative eigenvalue and three positive. That means that every point in space has a light cone associated with it, figure 4. These light cones can be tilted and change shape depending of the curvy aspect of the coordinates at each point.

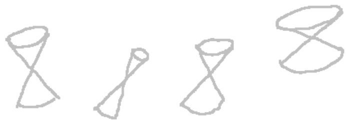

Figure 4 : Light cone at each point.

But at each point the metric has three positive and only one negative eigenvalue. And at each point there is the notion of time-like displacement, space-like displacement and light-like displacement.

So we have to make sure, when we write a metric, that it does have this property of the right number of positive and negative eigenvalues.

The property of having a certain number of eigenvalues positive and a certain number negative is called the *signature* of the metric. What was the signature of the metric of ordinary flat space ? Not ordinary space-time, just the page you are reading. It is $+\,+$. The signature of Minkowski space in special relativity with three spatial coordinates is $-+++$.

When somebody gives us a metric, or wherever we get it from – we might get it as a present in the mail, we might calculate it from some equations of motion, from some field equations – we should make sure that that metric has signature $-+++$. If it doesn't, it means we did something wrong. And not only should we have that signature at some point, but we should have it at every point.

Notice that the shape of the light cone, in particular its angle of openness, is a pure coordinate issue, see figure 4 and figure 2. In particular, if in the standard Minkowski metric and representation of figure 2, we chose units such that the speed of light is not 1 but the huge number c, the cone would be extremely flat, and the picture not very useful.

So much on the signature of the metric. Let's now move on to the metric of a massive body, and more specifically of a black hole. But first of all let's revisit geodesics, deriving them in another way.

Geodesics and Euler-Lagrange equations

We learned, in chapter 4, what is the definition of a geodesic. And we used the corresponding equation – equation (19) of chapter 4 – in the example of a free particle in a uniformly accelerated reference frame.

A geodesic is a curve whose tangent vector stays parallel to itself all along the curve. In other words, it is a trajectory where we always go straight.

In this chapter we are going to learn a different definition, which in many ways is more useful. But let us first recall the original definition,

$$\frac{d^2X^m}{d\tau^2} = -\Gamma^{\mu}_{\sigma\rho}\ \frac{dX^{\sigma}}{d\tau}\ \frac{dX^{\rho}}{d\tau} \tag{6}$$

The left hand side is the derivative of the tangent vector along some curve, which all along the curve should be equal to the double sum involving Christoffel symbols of the right hand side. That is the standard definition of a geodesic. But remember that in chapter 4, we also mentioned another : it is the analog of the definition of a geodesic in ordinary space, that is, the shortest distance between two points. Or better yet, it is the curve between two points whose length is stationary.

So another way of writing equation (6) is to say that we "extremelize" – or we make minimum – the length of the curve between two points.

Let's start with ordinary space, the page of this book, or a curved version of the page with hills and valleys. We take two points in the space and any curve between them. We calculate the distance along the curve.

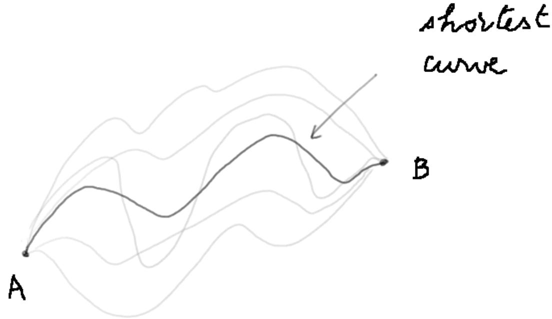

Figure 5 : Determination of a geodesic between A and B.
When the space is flat it is a straight line.

Then we search for the curve which minimizes its length.

How to calculate it ? Let's spell out *the logic*. We start, as we said, with any curve C between A and B. There are plenty of them shown in grey in figure 5. On the given curve C, for each little segment on the curve, we have

$$dS^2 = g_{mn}(X)\ dX^m\ dX^n \tag{7}$$

We are now in ordinary Riemannian space, not in space-time. We will come back to space-time in a moment. Equation (7) can be rewritten

$$dS = \sqrt{g_{mn}(X)\ dX^m\ dX^n} \tag{8}$$

This is just Pythagoras theorem applied to a little segment on curve C. Then we add them all up. This gives the distance along curve C in figure 5.

$$S = \int_{along\ curve\ C} \sqrt{g_{mn}(X)\ dX^m\ dX^n} \tag{9}$$

Finally we look for the curve C which makes S minimum or extremum. That's the logic. And by now we are familiar with *the maths* which implement this logic. We learned them in classical mechanics, in volume 1 of the collection *The Theoretical Minimum*. It is a problem in calculus of variation, analogous to minimizing the action of a particle along a trajectory.

In other words, we can think of equation (9) as expressing the action of a particle moving from A to B along curve C. Then the rule for calculating the geodesic is to "extremelize" this quantity, or to make it stationary. The equation that tells us how to minimize a quantity like S in equation (9) is called the Euler-Lagrange equation.

When you go from the principle of least action to the Euler-Lagrange equation, the principle of least action turns into a differential equation involving a lagrangian. Typically, when rewritten as explicitely as possible, the Euler-Lagrange equation becomes an equation of the type $F = ma$.

Going from minimizing the quantity in equation (9) to equation (6) is exactly that operation. In fact equation (6) looks like equating an acceleration to some thing. And the thing is a kind of force.

Now let's come back to relativity and to our actual problem of geodesic, where we are not concerned with ordinary distance but with proper time.

If we want to express the quantity to be minimized, which involves proper time, we deal with almost exactly the same expression as in equation (9) exept for a minus sign in front of the metric. From equation (4) we get that the proper time between point 1 and point 2 in space-time is given by

$$\tau = \int_1^2 \sqrt{-g_{\mu\nu}(X)\ dX^\mu\ dX^\nu} \tag{10}$$

This is the expression that we will want to minimize.

Let's suppose the expression defined by equation (10) really corresponds to the motion of a particle that starts at the space-time point 1 and ends at the space-time point 2, see figure 6.

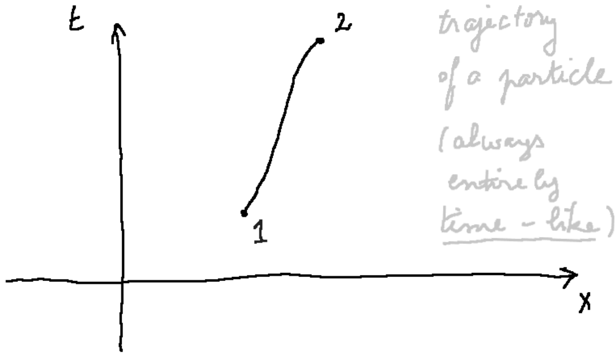

Figure 6 : Trajectory of a particle : geodesic in space-time.

The action we are interested in depends on one more quantity. It depends on the mass m of the particle. The actual action then is

$$A = -m \int_1^2 \sqrt{-g_{\mu\nu}(X)\ dX^\mu\ dX^\nu} \tag{11}$$

This is a definition of the mass. We will find out that putting a coefficient called mass here is important for thinking about energy and so forth. And the minus sign is strictly a convention in the definition of mass. We want to make this action A stationary.

What do we do with the right hand side of equation (11) ? A priori it is a completely unrecognizable object to work on with our mathematical toolbox. Okay, it is an integral. That is, it is just a sum of infinitely many infinitely small elements. [1] But where is the differential element ? What is

1. Mathematically speaking, it is more precisely *defined* as the

the variable to integrate over ? Etc.

Usually for us an integral which we know how to calculate, or at least manipulate, has the form

$$\int F(\ \textit{some variable}\ )\ d\ \textit{some variable}$$

where the variable may be some spatial quantity or may be time, or some other clearly identified physical quantity. Normally we don't see integrals where beneath the integral there is a square root, and inside the square root there is a product of differentials like $dX^\mu\ dX^\nu$.

Remember that we already met the same kind of integral in volume 3 on special relativity. We are going to solve it in the same way, introducing dt inside the integral, and even the square root, to arrive at a more familiar expression.

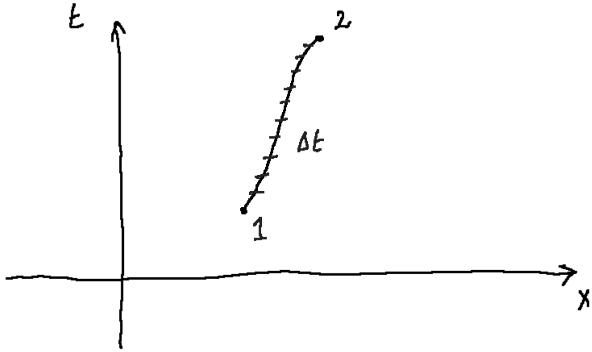

Figure 7 : Breaking the trajectory into little time segments.

limit, when the number N of elements goes to infinity and the sizes of the elements go to zero, of finite sums of N elements of the form $f(u_n)\Delta u_n$. Such a definition is also due to Riemann.

To start with, let's break up the trajectory of the particle into little time segments Δt, figure 7. Let's skip the finicky maths and make them directly dt, because, after a good basic course on calculus, we know that in our circumstances it is allowed. Equation (11) becomes

$$A = -m \int_1^2 \sqrt{-g_{\mu\nu}(X) \ \frac{dX^\mu}{dt} \ \frac{dX^\nu}{dt} \ dt^2} \tag{12}$$

Some of the differentials dX^μ are in fact dt, because t is one of the four coordinates in $X^\mu = (t,\ x,\ y,\ z)$. What happens when we have dt/dt? That is just 1. And what happens when we have dx/dt or the analog with y or z? It is just the ordinary velocity. We can also pull the dt^2 out of the square root, and obtain a standard differential element dt under the integral sign :

$$A = -m \int_1^2 \sqrt{-g_{\mu\nu}(X) \ \frac{dX^\mu}{dt} \ \frac{dX^\nu}{dt}} \ dt \tag{13}$$

At each time t, the quantity $\sqrt{-g_{\mu\nu}(X) \ \frac{dX^\mu}{dt} \ \frac{dX^\nu}{dt}}$, which is integrated over time, has a definite value along the trajectory. It is a certain function of the velocity and the position X. So we have transformed our expression (11) for the action into a conventional integral over time along the trajectory in figure 7.

The integrand in equation (13) is the lagrangian. It is the quantity which, in the calculation of a geodesic, plays exactly the same role as the lagrangian when we apply the principle of least action to calculate the trajectory of a particle in classical non-relativistic physics. Action by definition is equal to the integral of the lagrangian, which is itself a function of velocities and positions.

$$A = \int \mathcal{L}(\dot{X},\ X)\ dt \tag{14}$$

In summary, with our problem of making stationary the action given by equation (13), we return to a problem that we already met in classical mechanics, in volume 1 of the collection *The Theoretical Minimum*. How to calculate an equation of motion from an action ? In order to do that we solve the Euler-Lagrange equation (or equations) that the lagrangian must satisfy.

In our present problem, the lagrangian is

$$\mathcal{L} = -m\sqrt{-g_{\mu\nu}(X)\ \frac{dX^\mu}{dt}\ \frac{dX^\nu}{dt}} \tag{15}$$

Incidentally, is the quantity inside the square root positive ? [2] Can we take its square root, even though there is a minus sign in front of $g_{\mu\nu}$? Answer : yes, it is positive. The quantity $-g_{\mu\nu}(X)\ dX^\mu\ dX^\nu$ is the square of the proper time over a small element dX along the trajectory, see equation (1). It is always positive for a time-like trajectory. And particles always move on time-like trajectories. Real particles do not move faster than the speed of light.

In case the reader missed it, a space-like trajectory is one where the point moves faster than the speed of light.

2. Remember that when we deal with actions and lagrangians we are dealing with real quantities, for which the notion of maximizing or minimizing has a meaning. This would not be the case if we were dealing with complex numbers.

Let's recall what the Euler-Lagrange equations, that the lagrangian must satisfy, are. First of all we are going to partially differentiate $\mathcal{L}$ with respect to each of the variables $\dot{X}^\mu$.

But the first of these variables, which is the derivative of time with respect to time, is just one. So there is no correponding equation. We are only concerned with the three partial derivatives of $\mathcal{L}$ with respect to the components of ordinary velocity.

On the left hand side, for each of these partial derivatives, we take the derivative with respect to time, and equate it to the partial derivative of $\mathcal{L}$ with respect to the corresponding component of position.

Therefore the Euler-Lagrange equations are the three following equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}^m} = \frac{\partial \mathcal{L}}{\partial X^m} \tag{16}$$

where m runs from 1 to 3. We learned them in classical mechanics.

The point is : if you know the metric $g_{\mu\nu}(X)$, you can work out from equation (16) an equation of motion for the particle. And the particle's motion will be a geodesic in the sense of the trajectory with shortest proper time integrated along the curve.

How does this relate to the definition of a geodesic given by equation (6) ? Answer : if you work out exactly equation (16) with the given metric $g_{\mu\nu}(X)$, you will discover that you end up exactly with equation (6).

Exercise 1 : Given the metric $g_{\mu\nu}(X)$, show that the Euler-Lagrange equations (16), to minimize the proper time along a trajectory in space-time,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}^m} = \frac{\partial\mathcal{L}}{\partial X^m}$$

where $\mathcal{L}$ is given by equation (15), are equivalent to the definition of a geodesic given by equations (6), which say that the tangent vector to the trajectory in space-time stays constant,

$$\frac{d^2X^m}{d\tau^2} = -\Gamma^{\mu}_{\sigma\rho}\ \frac{dX^\sigma}{d\tau}\ \frac{dX^\rho}{d\tau}$$

It is generally true that to work with the action defined by equation (13), and with the Euler-Lagrange equations (16) that come from minimizing this action, is much easier than to work with a geodesic defined by equations (6).

We are going to do it in this lesson. We are going to work out some equations of motion for particles in a particular metric. The metric will be everybody's favorite metric : the Schwarzschild metric.

Schwarzschild metric

We now come to the problem of studying the metric and the motion of a particle in a real gravitational field, the gravitational field of the Sun, the Earth or a black hole or whatever you have, in other words a massive spherically symmetric object.

Figure 8 : Massive object and its gravitational field.

We are outside the mass of the object, far away from it, figure 8. And we are interested in the metric of space in there. So first of all we are going to write a formula for the metric and then we will check and see that this formula really does make sense.

By that we mean that, using the equations of motion (16), it would give rise to something that looks very familiar, namely Newton's equations for a particle moving in a gravitational field – at least when we are far away from the gravitating object, where the gravitation is fairly weak.

Here is the metric we are going to use. First of all if we were in a flat space, it would have the form

$$\tau^2 = dt^2 - \frac{1}{c^2}\, dX^2 \tag{17}$$

where henceforth dX^2 stands for $dx^2+dy^2+dz^2$. We expect indeed, as we go far away from the gravitating object, that the space-time there look flat.

But we know that the gravitating object does something to space-time. So equation (17) should not be quite right for the metric of the space-time created by the object. It should be right only in the limit when we are far away. Therefore we add a coefficient in front of dt^2, and we leave the part with dX^2 as it is.

$$\tau^2 = \left(1 + \frac{2U(X)}{c^2}\right) dt^2 - \frac{1}{c^2}\ dX^2 \tag{18}$$

$U(X)$ is the gravitational potential due to the object in figure 8.

Let's check that, as long as our particle is moving slowly, its geodesic equation, in the lagrangian form (16), just becomes Newton's equation for a particle moving in a gravitational potential $U(X)$.

The general form of the lagrangian is given by equation (15), where inside the square root is just $d\tau^2$. Now that we know the metric, from equation (18), we can write the lagrangian or the action more explicitely

$$A = -mc^2 \int \sqrt{\left(1 + \frac{2U(X)}{c^2}\right) dt^2 - \frac{1}{c^2}\ \frac{dX^2}{dt^2}\ dt^2} \tag{19}$$

Noting that $\frac{dX^2}{dt^2}$ is $\dot{X}^2$, and doing a bit of cleaning, this is the same as

$$A = -mc^2 \int \sqrt{\left(1 + \frac{2U(X)}{c^2}\right) - \frac{\dot{X}^2}{c^2}}\ dt \tag{20}$$

When we work with c explicitely present in the formulas, the expression for the action must carry a c^2 next to the mass. We are familiar with that. It can be derived from reasoning on dimensions, or we can remember that the action has units of energy multiplied by time. Recall the most basic action in classical mechanics : $A = \int \frac{1}{2} m v^2 dt$.

We could work with the lagrangian, in equation (20), as it is, but what we are interested in is what happens when we move with slow speed relative to the speed of light. So the next step is to approximate the lagrangian, and look at the Euler-Lagrange equations with the approximate lagrangian [3].

We are interested in slow motion because we want to show that equation (20) really does give rise to Newton's equations in the non-relativistic limit. The non-relativistic limit is the one where c is taken to be very large.

We could be tempted to just erase anything with a $1/c^2$ beneath the integral sign, but – boy ! – this would kill just about everything in equation (20).

Inside the square root we can reorgonize terms as one plus a small quantity :

$$\sqrt{1 + \frac{1}{c^2}\left(2U - \dot{X}^2\right)} \tag{21}$$

3. As usual, we rely on the mathematical fact that, in this case, it is okay to approximate the lagrangian first and then to solve the Euler-Lagrange equations rather than solve properly the Euler-Lagrange equations first and then look at the limit case when $\dot{X}$ is small. In other words, we can invert two big operations.

Next we use the binomial theorem :

$$\sqrt{1+\epsilon} \approx 1 + \frac{\epsilon}{2}$$

So expression (21) can be approximated by

$$1 + \frac{1}{2c^2}\left(2U - \dot{X}^2\right) \tag{22}$$

Going back to the expression of the action, in equation (20), we obtain

$$A = \int \left(-mc^2 - mU + \frac{m}{2}\dot{X}^2\right) dt \tag{23}$$

Inside the integral, we have the lagrangian when the speed of the particle is small, that is, when we can make a non-relativistic approximation.

If we use this lagrangian in the Euler-Lagrange equations, the constant term $-mc^2$ has no effect. The only thing we do with a lagrangian is differentiate it. When we differentiate a constant we get zero. So we can disregard that term.

The other two terms in the lagrangian are a conventional kinetic energy, $\frac{m}{2}\dot{X}^2$, minus a potential energy, $mU(X)$, which actually depends on X. For the potential energy we get to choose the function we like. Incidentally in a gravitational problem the potential energy of a particle is always proportional to its mass.

Finally, when we use this lagrangian, the Euler-Lagrange equations will of course simply produce Newton's equation for a particle in a gravitational field $U(X)$, just as they did with exactly the same calculations in classical mechanics.

The equation will be

$$m\ddot{X} = -m\frac{\partial U}{\partial X} \tag{24}$$

The term on the right hand side is a force. On the left hand side is an acceleration multiplied by the mass. And the mass cancels.

The main point here was that the action given by equation (13), which is equivalent to the geodesic given by equation (6), is easily worked out just by thinking of the Euler-Lagrange equations.

And it is even easier to work out if you are in the non-relativistic limit, where you just say c is very large, $1/c^2$ is very small, and just expand the square root in the expression of the action.

But the important point that we learned in doing the above calculation is that in some first approximation – or perhaps no approximation – we can write

$$-g_{00} = \left(1 + \frac{2U(X)}{c^2}\right) \tag{25}$$

Of course in the equation (18) we wrote for τ^2, there could be even smaller terms than $1/c^2$, terms in $1/c^4$ or $1/c^6$, etc. But they would not be important for in non-relativistic limit.

So we cannot say with complete confidence that $-g_{00}$ is one plus twice the potential energy of a particle divided by c^2.

But we can say that it must be true to the first order in small quantities – small quantities meaning quantities with one over c^2.

Now for the gravitational potential energy created in space by a body of mass M, see figure 8, we shall use

$$U(X) = -\frac{MG}{r} \tag{26}$$

where G is Newton's constant, and r is the distance away for the center of the body.

Now we can write down our first guess at the metric of space-time surrounding a gravitational mass in figure 8. Here it is :

$$d\tau^2 = \left(1 - \frac{2MG}{c^2\ r}\right) dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) + ... \tag{27}$$

where $r = \sqrt{dx^2 + dy^2 + dz^2}$. And the ... after the spatial terms stand for smaller things, that is things one or more orders of magnitude smaller than $1/c^2$.

Is that beginning to look familiar ? Readers who have some familiarity with cosmology may recognize the Schwarzschild metric, o the metric of a black hole – but not quite.

Next, let's consider the term

$$dx^2 + dy^2 + dz^2 \tag{28}$$

It is the ordinary metric of three-dimensional space. In equation (27) defining the metric of space-time created by a massive body, only the time-time component of the metric has been fiddled with. The space-space components of the metric and everything else have not yet been fiddled with. They will in a moment but not too much.

The space-space components in equation (27) are the metric of the ordinary flat space. Let's take flat space in three dimensional polar coordinates. These coordinates are characterized by a radius, namely the distance from the center of the Sun – if we think of the body in figure 8 as the Sun –, and a pair of angles. They can be a polar angle and an azimuthal angle.

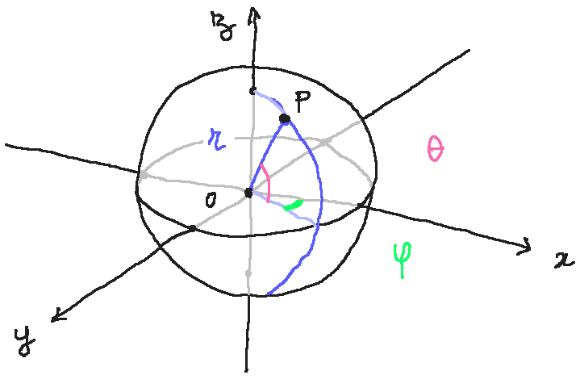

Figure 9 : Spherical polar coordinates.

r is the distance from the Sun O, to P. The angle ϕ can be taken from the pole or from the equator. We choose to measure it from the equator. It is also called the latitude. And ϕ, the azimuthal angle, is also called the longitude.

How is the ordinary 3D Euclidean metric $dx^2 + dy^2 + dz^2$ expressed in spherical polar coordinates ? The formula, that you may have learned in trigonometry at the end of high school, is

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 \ (d\theta^2 + cos^2\theta \ d\phi^2) \qquad (29)$$

Despite its complicated look, it is simply the square of the length element on the surface of the sphere, $r^2 \ (d\theta^2 + cos^2\theta \ d\phi^2)$ that we have already met, to which is added dr^2 to complete Pythagoras theorem in three dimensions.

Why are we considering polar coordinates ? Answer : simply because polar coordinates are the good coordinates for studying the central force problem. We don't want to use x, y and z to study the motion of a particle in a gravitational field. We want to use the polar coordinates. So we are going to write the metric given by equation (27) in terms of polar coordinates.

We are going to make one change : we will give $d\theta^2 + cos^2\theta \ d\phi^2$ a name, so we don't have to write it over and over. We call it $d\Omega^2$. There is no other reason than to avoid writing it all the time in its full expression with θ and ϕ. Equation (27) can be rewritten

$$d\tau^2 = \left(1 - \frac{2MG}{c^2 \ r}\right) dt^2 - \frac{1}{c^2} \ dr^2 - \frac{1}{c^2} \ r^2 \ d\Omega^2 + ... \qquad (30)$$

The variable r is just the radial distance. And $d\Omega^2$ is the metric on the surface of a sphere. If you keep r fixed, the term $r^2 \ d\Omega^2$ is the contribution to the space-time metric when you move some angles θ and ϕ on the sphere.

Now if we look at this metric, given by equations (30) or (27), there is something terribly wrong. It is fine when we are far away. But there is something deeply wrong when we get close to the center.

What happens when r becomes very small ? At some point $2MG/c^2r$ becomes bigger than one, and the coefficient

$$\big(1 - 2MG/c^2r\big)$$

becomes negative. In other words, the time-time part of the square of the proper time $d\tau$, the coefficient $-g_{00}$, becomes negative.

Since the other terms – the space-space parts – in the definition of $d\tau^2$ given by equations (30) or (27) are also negative, we run into a problem. The metric tensor or matrix $g_{\mu\nu}(X)$ then only has positive terms on its diagonal (and the terms off diagonal are all zero). Therefore it has four positive eigenvalues instead of one negative eigenvalue and three positive eigenvalues.

Where does the problem arise ? It arises when $\big(1 - 2MG/c^2r\big)$ changes sign, that is when

$$r \geq \frac{2MG}{c^2} \tag{31}$$

The quantity $2MG/c^2$ is just some positive number, specific of the body we are looking at in figure 8. And the metric no longer has three positive eigenvalues and one negative.

Of course this happens if all the mass of the body is somehow point-like or almost point-like [4]. This leads us to the topic of black holes.

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4. As the reader knows, if we enter inside the Earth, we will only

Black holes

A black hole is a massive body whose mass M is *extremely* concentrated in a very small sphere. It is concentrated in a sphere of radius smaller than the value $2MG/c^2$ which we discovered to be causing a problem in the metric.

$2MG/c^2$ is a distance characteristic of the black hole. It is called the Schwarzschild radius of the black hole.

When we get closer and closer to the black hole, at some point, when we cross the Schwarzschild radius, the coefficient $-g_{00}$ changes sign and becomes negative, or equivalently g_{00} becomes positive, and the whole metric of equation (30) now has four positive eigenvalues.

The meaning of that is we somehow passed into a region where there are four space directions and no time direction. That is bad! That is something we don't want. It is not a good thing.

What really happens? When we study the field equations of general relativity – which we are going to do – we will discover really what happens. For the time being let me just say that at the point where $\left(1 - 2MG/c^2r\right)$ changes sign, the rest of equation (30) will also change. We will put an extra coefficient in front of the term dr^2/c^2 in equation (30) to the effect that when the time-time component of the tensor changes sign, the r-r component will also change sign.

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be submitted to the gravitational field created by the mass at radius shorter than where we are, the mass farther away than us, all around the Earth, will play no role.

So what happens ? The term $\left(1 - 2MG/c^2 r\right) dt^2$ turns into a space direction, but the term with dr^2 turns into a time direction. The signature of the metric is maintained : one time-like direction and three space-like directions.

What can we put, in equation (30), in front of the dr^2/c^2 to make it flip sign, when the time-time component flips sign ? Well, we could also put in $\left(1 - 2MG/c^2 r\right)$ in front of the dr^2/c^2. So if we cross the Schwarzschild radius the first term changes sign and the dr^2 term changes signs too. But that turns out not to be quite the right thing.

Einstein's field equations give us a different answer. Instead of using the same coefficient as in front of dt^2, use its inverse. From now on disregarding the smaller terms in $1/c^4$ or more, the metric becomes

$$d\tau^2 = \left(1 - \frac{2MG}{c^2\ r}\right) dt^2 - \left(\frac{1}{1 - \frac{2MG}{c^2\ r}}\right) \frac{1}{c^2}\ dr^2 - \frac{1}{c^2}\ r^2\ d\Omega^2 \tag{32}$$

When we cross the Schwarzschild radius we do have both coefficients changing sign at the same time. Therefore the signature $-+++$ of the metric is preserved. But something very odd happens : the thing that we are calling t becomes a space-like direction, and the thing that were calling r becomes a time-like direction. They flip.

This is not easy to visualize but I will give you the tools to visualize it as we go along. Something happens as we cross a certain threshold – the surface of a sphere of radius r equals $2MG/c^2$. The radius r becomes a time variable, and the time t becomes a space variable. This is completely mysterious for the moment, but we will explain in some geometric detail what is happening at the Schwarzschild radius.

We will discover that, when we are moving radially toward the black hole, to cross the Schwarzschild radius takes an infinite amount of coordinate time t but a finite amount of proper time τ. In other words, somebody falling, with a wristwatch, will say it takes him or her a finite amount of time to cross that threshold. But somebody, watching from the outside that person fall through, will say it takes an infinite amount of time. That is a characteristic of the metric given by equation (32). And that is one of the things we want to work out in this lesson.

Event horizon of a black hole

Let's think about where is happening this peculiar phenomenon of time and space switch. It is happening at coordinate $r = 2MG/c^2$. The speed of light c is a huge number, and its square is in the denominator. So it is a very small distance, but only if M itself is not huge because the Schwarzschild radius depends on the mass of the gravitating object. For the Earth, if its mass were concentrated in a smaller distance than that, this distance would be about a centimeter.

When we cross the Schwarzschild radius, the reader may wonder what happens to the wristwatch. Does it begin to record spatial distance ? Answer : no. It keeps ticking in a time direction. Clocks don't care about what coordinates we use.

We are going to discover that the phenomenon of time and space switch is an artifact of our coordinate system. There is nothing special going on at the Schwarzschild radius. We will see that $(t,\ r,\ \theta,\ \phi)$ are just awkward coordinates that make it look like something funny is happening at $r = 2GM/c^2$.

In truth there is nothing funny happening at the Schwarzschild radius. We are going to work out and see exactly why. But for the moment it does look like there is something going on there. Looking at the metric tensor expressed in the coordinates $(t,\ r,\ \theta,\ \phi)$, when the g_{tt} coefficient vanishes, the g_{rr} coefficient becomes infinite. It looks like some terrible things happen to the geometry.

But in fact the geometry is completely smooth over the threshold $r = 2GM/c^2$. There is nothing special happening. The light cones are perfectly healthy all over the region around the Schwarzschild radius. There is nothing special going on. But we have to work harder to see it.

Here is some historical note. Einstein submitted his field equations to the Prussian Academy of Science on November, 25 1915. Then Schwarzschild [5] studied Einstein's paper and, in December 1915, while fighting on the Russian front during WWI, wrote down equation (32) for the metric of a gravitating body (outside the body). He unfortunately died shortly afterwards. What did Einstein already know of it ? He knew about the first term in front of dt^2 and he knew

5. Karl Siegmund Schwarzschild (1873-1916), German astrophysicist.

part of the second term in front of dr^2 in equation (32).

To understand what we mean, let's examine this second term when r is not too small, that is when $2MG/c^2r$ *is* small. Using our favorite tool to make approximations – the binomial theorem –, the coefficient in front of dr^2, in equation (32), can be expanded as follows

$$\frac{1}{1-\frac{2MG}{c^2\ r}} = 1 + \frac{2MG}{c^2\ r} + \left(\frac{2MG}{c^2\ r}\right)^2 + ... \tag{33}$$

On the right hand side the first term after 1 has $1/c^2$, the second term $1/c^4$, etc. In other words, what we are doing, in equation (32), is correcting the dr^2/c^2 of equation (30) with corrections that are very small – of the order of magnitude of $1/c^4$ or smaller. They contribute to the motion of the particles, but not in the non-relativistic limit.

Let's close this historical note, and come back to the metric of equation (32). Before we analyze it and see what is going on at a point where we cross the horizon of the black hole – another name for the surrounding sphere of Schwarzschild radius –, let's use it and see if we can figure out how particles move in the presence of this metric, not in the non-relativistic limit but without taking any approximation. It is not that hard.

By the way, if we are not going to do any approximation, we might as well set c equal to one. It is just a choice of units. Then the metric looks like this

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(\frac{1}{1-\frac{2MG}{r}}\right) dr^2 - r^2 d\Omega^2 \tag{34}$$

Two kinds of trajectories are easy to study : circular orbits – they reserve some surprise if you do them –, and radial trajectories. In this lesson, let's study a radial trajectory, a spacecraft falling along a straight line and a priori going to crash on the black hole.

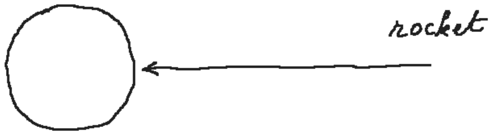

Figure 10 : Radial trajectory of a spacecraft falling toward the black hole.

The question is : how long does it take for the spacecraft to reach the horizon ? We are going to measure this duration from two points of view : that of an external observer, using coordinate time t, and that of an astronaut in the spacecraft, using the proper time τ of the spacecraft and himself. The results are very different.

The main point of these calculations is to show you how to work with the metric of equation (34).

When you have to do any such calculation, you always start with the lagrangian for the particle. Then you work out the Euler-Lagrange equations or whatever you need to do with the lagrangian. And you solve the equations. There are tricks you can do. I'm going to show you some of the tricks that allow you to do the problem.

So we go back to the lagrangian for the in-falling particle. Remember that particles follow geodesics in space-time, so, from equation (11), the lagrangian is just $-m\sqrt{d\tau^2}$. We get $d\tau^2$ from equation (34). Assuming that the spacecraft goes straight and that Ω doesn't change, we can omit the term $r^2 d\Omega^2$. Therefore the action, that must be stationary, is

$$A = -m \int \sqrt{\left(1 - \frac{2MG}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{2MG}{r}}\right) dr^2}$$

As usual, to make sense out of this integral, below the integral sign we divide inside the square root by dt^2, and multiply outside the square root by dt.

$$A = -m \int \sqrt{\left(1 - \frac{2MG}{r}\right) - \left(\frac{1}{1 - \frac{2MG}{r}}\right) \frac{dr^2}{dt^2}} \; dt$$

The ratio dr^2/dt^2 is just the radial velocity squared, that is $\dot{r}^2$. So we got our lagrangian

$$\mathcal{L} = -m \sqrt{\left(1 - \frac{2MG}{r}\right) - \left(\frac{1}{1 - \frac{2MG}{r}}\right) \dot{r}^2} \tag{35}$$

The coordinate in this case is r. And this is $\mathcal{L}(\dot{r},\ r)$.

Fortunately applying the Euler-Lagrange equations to compute the trajectory is easy. Or at least it is easy to see what is happening at the Schwarzschild radius. Let's not worry about the exact solutions. Exactly solving the problem is a little bit hard. But seeing what happens is easy.

There is always a conserved quantity. What is the conserved quantity that is always there ? Energy.

What is energy in terms of the lagrangian ? Remember from volume 1, chapter 6, of the collection *The Theoretical Minimum*, on the least action principle :

the energy is the hamiltonian.

And the hamiltonian is expressed in terms of the lagrangian.

To compute the hamiltonian, the first thing we do is calculate the generalized conjugate momentum to r. It is

$$p = \frac{\partial \mathcal{L}}{\partial \dot{r}}$$

In the general case, when instead of one coordinate r we have a collection of coordinates q_i and their conjugates p_i, the general formula for the hamiltonian is [6] :

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}$$

In our case it becomes simply

$$H = p\dot{r} - \mathcal{L}$$

The calculations are left to the reader. The result is

$$H = \frac{m\ (1 - 2MG/r)}{\sqrt{(1 - 2MG/r) - \dot{r}/(1 - 2MG/r)}} \tag{36}$$

6. It is equation (4) of chapter 8 of volume 1.

It is a complicated ugly expression. But it is a definite thing. What does it depend on ? It depends on the mass m of the particle or spacecraft, the distance r to the black hole center, and the velocity $\dot{r}$. It is the energy.

We know that this energy is conserved over time. We can call it E instead of H, and it does not change with time.

Then equation (36), giving the energy, enables us to express $\dot{r}$ as a function of that energy E. With a little bit of algebra we get

$$\dot{r}^2 = (1 - 2MG/r)^2 - \frac{(1 - 2MG/r)^3}{E^2} \tag{37}$$

This expression again looks unwieldy. But it doesn't really matter. What is important is that we can easily see what happens when r gets near $2MG$, that is, when the spacecraft gets close to the horizon of the black hole.

Equation (37) tells us that as r gets near $2MG$, in other words as the spacecraft approaches the horizon of the black hole, its velocity slows down to zero. Contrary to intuition, when the spacecraft falls towards the Schwarzschild radius, instead of accelerating, its velocity gets smaller and smaller.

Exercise 2 : Show that from equation (36) for the energy, and equation (37) for $\dot{r}^2$, it follows that

$$\dot{r} \approx \sqrt{\frac{r - 2MG}{2MG}} \tag{38}$$

Equation (38) offers another way to see that as r approaches $2MG$ the spacecraft decelerates. It is a surprise because we might think that the thing would accelerate like, in elementary physics, would do a particle falling toward a massive body. But instead it slows down [7].

The speed gets asymptotically to zero, and the spacecraft or particle itself never passes the Schwarzschild radius. You might say figuratively that it takes forever to reach the horizon.

Let's go back to the Schwarzschild metric given by equation (34). Let's write it again below leaving aside the $d\Omega^2$ piece which does not play any role

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{2MG}{r}}\right) dr^2 \qquad (39)$$

How does a radial light-ray move ? Think of a light-ray beamed from some point either toward the center of the black hole or moving away, it doesn't matter for our analysis : it moves radially.

7. Notice that we still are in classical physics in the sense of not quantum. The only special things from elementary physics are

1. particles move along geodesics in space-time
2. we look at a massive body whose mass is *extremely* concentrated
3. we look at what happens at the Schwarzschild radius, which is ordinarily very small (recall that it is $2MG/c^2$)

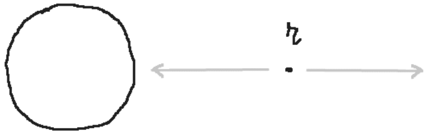

Figure 11 : Radial light-ray beamed from r in each direction.

The trajectory of the light-ray is a solution to equation (38) that says that it is light-like. Light-like means that $d\tau^2 = 0$. So a light-ray satisfies

$$\left(1 - \frac{2MG}{r}\right)^2 dt^2 = dr^2$$

or equivalently

$$\frac{dr}{dt} = \pm \left(1 - \frac{2MG}{r}\right) \tag{40}$$

That is a light-ray. It is the fastest thing that can ever be. When r is big the right hand side of equation (40) is almost 1, that is the speed of light in our units. So there is nothing troublesome there. But as r decreases to the Schwarzschild radius, we see that the magnitude of the radial velocity, which is $(r - 2MG)/r$, goes to zero.

In other words, even a light-ray somehow gets stuck as it approaches the black hole horizon !

Is it moving with the speed of light ? Answer : Of course it is moving with the speed of light. What else can a light-ray do? But the speed of light has this property that *in the particular set of coordinates* $(t,\ r,\ \theta,\ \phi)$, the speed of light *measured by dr/dt* goes to zero as you get closer and closer to the Schwarzschild radius. Therefore nothing even including a light-ray can get past the surface of the horizon... or so it seems.

The velocity goes to zero at the horizon, whether we are incoming or outgoing. So wherever we are, something odd happens beyond the horizon. And we are going to work that out.

Now let's turn to someone moving with the spacecraft of figure 10, or with a photon of figure 11. From the point of view of a person falling in, time is the proper time τ. He or she will just go sailing right through the Schwarzschild radius. The phenomenon of dr/dt going to zero is an artifact at $r = 2MG/c^2$ of the peculiar spherical polar coordinates used for the stationary frame. There is nothing really going on that is strange at that distance. And the person will reach the black hole in a finite amount of proper time.

We will see this phenomenon in more details in chapter 6. In particular we will see the relation between stationary time t and proper time τ. Equation (39) shows that when r approaches the Schwarzschild distance, the coefficient $(1 - 2MG/r)$ goes to zero. When the coefficient is going to zero, a given amount of dt corresponds to a smaller and smaller amount of proper time $d\tau$. As a consequence, proper time in some sense "slows down". That is why something can take an infinite amount of time in one frame (the frame where time is t), and a finite amount of time in the other frame (the frame where time is τ).

One last point that we should emphasize is that if we were really talking about the Sun or the Earth, these bodies are not compact enough to be black holes. A black hole is a body whose mass is contained within its Schwarzschild radius. The figure is about 3 kilometers for the Sun, and about 9 millimeters for the Earth. However the Earth radius is 6000 kilometers.

Equation (34) for the metric, or equation (37) for $\dot{r}$ are only valid *outside* the gravitating body itself. It is only if the Earth somehow would collapse within a radius of 9 millimeters, keeping its mass, that there would be a Schwarzschild radius to speak about, leading to dr/dt going to zero and so forth. Same thing for the Sun : only if it collapsed within a sphere less than 3 kilometers in radius would it have a black hole horizon.

Inside the gravitating body, the metric is different from equation (34). And nothing bizarre happens in the center of the Earth. The center of the Earth does not have a black hole horizon.

To have the complete metric in space-time created by a gravitating body you have to piece together the metric outside and the metric inside the body.

Black holes do have peculiarities. But they are somewhat different from what popular science says. For example, in popular science, it is said that we can throw things – tissues, cars, planets, anything – into a black hole, and make it grow. But I showed you that these things thrown at the black hole will never get there. So there is a problem.

Next chapter will be devoted to deepening our knowledge of black holes.