Lesson 6 : Black holes

Notes from Prof. Susskind video lectures publicly available on YouTube

Schwarzschild metric

Last chapter, we began discussing the properties of a Schwarzschild black hole. Let's review quickly what we did, before going on.

The Schwarzschild solution is an idealized solution in the sense that the Newtonian force law

$$F = -\frac{mMG}{r^2} \tag{1}$$

is an idealization under the assumption that the mass creating the gravitational field is a point at the center of coordinates.

If the mass is spread out then of course we don't really believe equation (1) to hold in the interior of where there is mass. And in the interior of the Earth equation (1) is not correct.

But is true in the exterior, out beyond the surface of the Earth at least if we ignore the atmosphere and the other forms of gravitating mass.

The point mass in Newtonian mechanics is an idealization because you never really ever reach the volume of a point mass. Real materials have a certain stiffness. You can compress them but you can never squeeze them to infinitely small radius. They just have a certain resistance to being squeezed.

And gravity is only so strong, even though the gravitational field appears to get very very large at short distances.

The nature of the material is always such that if you squeezed it down to arbitrarily small distance, it was spring back.

This is not so in the general theory of relativity. We are going to see an example where, if matter gets too close to the singularity at $r = 0$, it gets sucked in a way that doesn't happen in Newtonian gravity.

For example in Newtonian physics, if we have a force center, and we shoot a particle with infinite precision along a radius toward the force center, it will indeed hit the center. But if we have the littlest deviation away from perfection and we don't quite – even with a very small impulsion – shoot in a radial direction, the particle will not hit the center. It will swing around it as in figure 1.

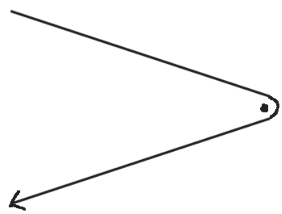

Figure 1 : Shooting a particle toward a force center without perfect precision, in Newtonian physics.

Effectively there is a kind of force repelling the particle from the center. It is a centrifugal force. If your aim is not perfect, the particle will have a little bit of angular momentum relative to the center, and the effective centrifugal force will keep it from going to the center.

So in Newtonian mechanics, even though the center pulls very hard when you get close to it, nevertheless it is infinitely unlikely for an in-falling point particle to actually hit the center.

The situation is different in the general theory of relativity. Gravity is even stronger. It is so strong that it overwhelms the centrifugal barrier, and pulls anything into the center.

Let's now go back to the Schwarzschild metric that we began to study in the preceding chapter. Notice that we did not derive the metric. We just wrote it with partial heuristic justification. Later in the book, when we establish Einstein's field equations in chapter 9, we will show that the Schwarzschild metric is a solution to them. But so far for us it is a specific metric that is a given, a specific geometry, and we are going to analyze the consequences of the metric. The metric is given by this equation

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{2MG}{r}}\right) dr^2 - r^2 d\Omega^2 \quad (2)$$

Remember that $d\Omega^2$ is just the ordinary metric on the unit sphere expressed with two spherical angles, the angle θ from the equator and the azimuthal angle ϕ (see figure 9 of chapter 5).

$$d\Omega^2 = d\theta^2 + cos^2\theta \; d\phi^2 \quad (3)$$

Remember also that as r crosses the value $2MG$ the coefficient in front of dt^2, in equation (2), changes sign, but so does the coefficient in front of dr^2, therefore the signature of the metric dS^2 (which is the opposite of $d\tau^2$) remains one negative sign and three positive signs.

Regarding the two spherical angles, θ and ϕ, sometimes we may be interested in only one of them. For instance, when we study an orbiting particle, we are going to locate it in space-time with t and ϕ. Radius r will remain constant and $\theta = 0$. (We could also fix ϕ and consider θ as the variable angle of the orbiting particle.)

Schwarzschild radius or black hole event horizon

Let's draw a picture of the metric, with the time axis t vertical, and two spatial axes x and y forming a horizontal plane at $t = 0$, figure 2. Of course there should be three spatial axes, x, y and z, and the plane at $t = 0$ should really be a 3D space, but it is not possible to draw such a thing, together with the time axis, on a page.

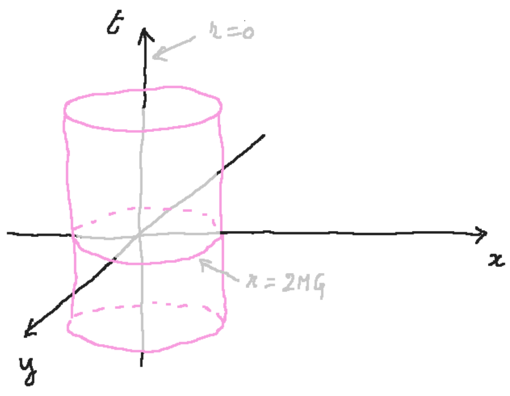

Figure 2 : Space-time in the vicinity of a black hole.

The radius $r = 2MG$, corresponding to a cylinder in figure 2, is called the *Schwarzschild radius*, or the *event horizon* of the black hole. We will understand why as we go along.

Let's look at what is going on very far away. When r is very big, the coefficient in front of dt^2 becomes almost 1, and the coefficient in front of dr^2 almost -1. Then it is just flat space-time in polar coordinates.

But as we move into $r = 2MG$ something happens and the geometry is characteristically different. Something looks bad at that value of r. First of all the coefficient of dt^2 becomes zero. But even worse the coefficient of dr^2 becomes infinite. So something is blowing up in the maths. However it is only a consequence of the mathematical coordinates $(t,\ r,\ \theta,\ \phi)$ chosen to chart space-time. We are going to see that nothing of any real physical significance is happening at that point.

On the other hand at $r = 0$ there is also a problem in the equation, and this time is correspond ta a singularity in the physics as well. The coefficient in front of dt^2 becomes infinite, and that in front of dr^2 zero. What is going on at $r = 0$ is that the curvature is becoming infinite.

At the Schwarzschild radius, $r = 2MG$, no large curvature or enormous deviation from flat space is observed. But at $r = 0$ all hell breaks loose. If we calculated the curvature tensor to find out how curved the geometry is at the center, we would find that all of its components become infinite.

There is another way to describe that phenomenon. Curvature means tidal forces. Anything falling in, when it would hit $r = 0$, would experience infinitely strong tidal forces and be torn to pieces. It would be stretched radially and squeezed in the angular direction an infinite amount. So $r = 0$ is the real place where serious disaster occurs – not the event horizon.

We talked a little bit about radially in-falling things. Let's go over what the conclusion was. We won't need to go into the maths again, because we are going to see exactly what is going on in terms of pictures. We will see that it takes an infinite time for something falling in to go past $r = 2MG$.

In figure 2, the sufaces of constant time t are the horizontal planes at different height. Something falling in, say along the x-axis, would have a trajectory shown in figure 3.

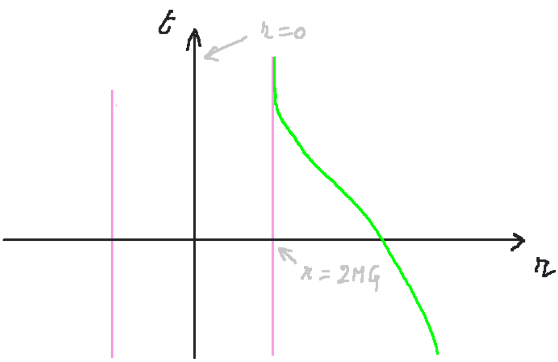

Figure 3 : Trajectory of a particle falling toward a black hole, in spherical polar coordinates (with $\theta = \phi = 0$).

The particle or object accelerates for a while. But as it gets closer and closer to the horizon, its trajectory becomes asymptotic to the vertical axis $r = 2MG$, while never quite getting there.

If the object is a meter stick falling in the long way, its front end and back end would follow trajectories as shown in figure 4.

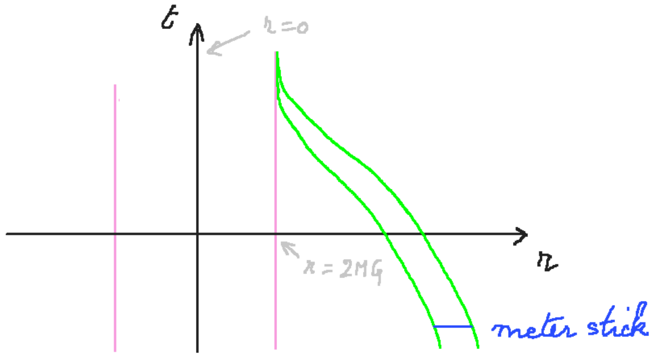

Figure 4 : Meter stick falling in radially.

The only way to make sense out of this phenomenon is to say, *at least in these coordinates*, that the gap in r between the front end and the back end of the meter stick shrinks to zero.

It is actually a form of Lorentz contraction. As something falls in, even though it appears to be slowing down, its momentum is actually increasing, and it is Lorentz contracting.

Think also about a clock. Let's imagine that the meter stick was a clock. How do we build a clock that is also a meter stick? Well, we can put two mirrors at either end of the meter stick and allow a light beam to go back and forth, and simply count the number of reflexions.

But clocks as they move – whatever motion they have – don't tick off time t of equation (2). They tick off proper time. That is the meaning of the proper time : it is the ticking of a clock moving together with the moving object under study.

Time t of equation (2) is called *coordinate time*. It is just the coordinate that we are using to describe the entire system.

Notice that if we are very far away, that is if r is very big, and if we are not moving, then $d\tau$ is equal to dt, at least to a high approximation, because $1 - 2MG/r$ is essentially 1, and dr and $d\Omega$ are 0. So very far away, proper time and ordinary time are the same.

But as we get closer and closer in, for a given amount of dt, the incremental proper time $d\tau$ gets smaller and smaller. For example, consider the coordinate time elapsing from t_1 to t_2. Think of t_1 and t_2 as horizontal lines in figure 4. How much proper time elapses *along the trajectory* of the object between these two horizontal lines ? When t_2 increases by dt, the corresponding proper time increment $d\tau$ gets smaller and smaller, as the value of t_2 gets bigger.

So, while it appears that the amount of t it takes to fall into the black hole is infinite, we could ask how much τ has elapsed by the time an in-falling object eventually arrives at the at the horizon. And that proper time is finite. Again we are going to see this in pictures, without doing any mathematics or calculations. We will see it just from simple diagrams.

Consider somebody falling in with a clock. Will that person experience an infinite amount of time before he or she gets to the horizon, or a finite amount ? By the way, the clock doesn't have to be an mechanical device. By "experience" we mean the time the person feels. That can be heart beats, or any other physiological process that is also a time-keeper. The answer is : they experience the proper time. So the question is : how much proper time does it take to reach the horizon ?

To answer the question, we could solve equations, we could calculate, do integrals. But fortunately we don't really have to. Diagrams will take care of it.

Before we get to the diagrams, let's solve another problem which does not have to do with somebody falling into a black hole, but with another question : orbiting the black hole.

Light-ray orbiting a black hole

Orbiting a black hole means going around it as time t goes forward. If we think about a circular orbit, in the spacetime representation of figure 2, the trajectory will have the shape of corkscrew (a usual "right" corkscrew, or a "left" corkscrew).

If we don't try to represent the coordinate time t axis, we can figure out the trajectory as in figure 5.

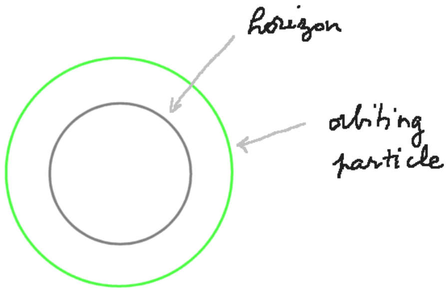

Figure 5 : Particle orbiting a black hole.

The center has coordinate $r = 0$, the horizon $r = 2MG$, and the particle r equals some larger value.

In fact we won't study the trajectory of any particle, we shall study the trajectory of a light-ray. As we know from Einstein and from many other sources, a light-ray passing by a massive body is bent.

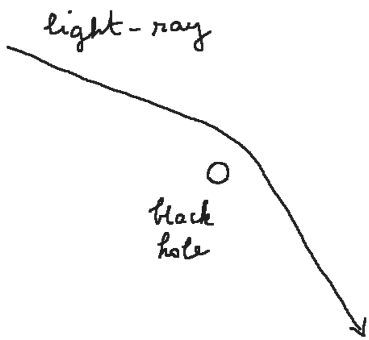

Figure 6 : Light-ray trajectory bent by a black hole.

In chapter 1 we saw that in Newtonian physics a uniformly accelerated elevator has an effective gravitational field inside it. From that, we infered the *equivalence principle* : accelerated frames and gravitation fields are equivalent [1].

Secondly we saw that a light-ray crossing the elevator appears bent to someone in the elevator, so we deduce that real gravity should bend light-rays. Then in chapter 4, we saw that in special relativity again, a uniformly accelerated frame of reference – which was something trickier to define – shows an effective gravitational field.

In figure 6 the same thing is happening : gravity bends trajectories. And obviously as the light-ray passes closer to the black hole it will be bent more. Why ? Because gravity becomes stronger.

As we are moving toward the Schwarzschild radius, the light-ray really whips around the black hole. And at some point – it is not at the horizon, it is further out than the horizon – there exist trajectories which will just go in circle around the gravitating object.

By the way, this will not happen near the surface of the Earth. You would have to squeeze the Earth to a small enough volume to transform it into a black hole. We already said that this would be a ball – with the same mass – of radius less than 9 millemeters.

So we are not going to see light-rays circling around the Earth. Nevertheless it is very interesting to try to understand orbiting light-rays around black holes.

1. at least locally when the curvature of a real gravitational field can be neglected.

The next step is a little bit calculation-intensive. I'm going to show you the sequence of operations to calculate the circular orbits. We are going to do it using the rules of classical mechanics that we learned in volume 1 of the collection *The Theoretical Minimum*. The reason we will do it this way is because it is the only easy way, and even then it is not so easy.

An important point to note is that the things we learned in classical mechanics are not disconnected from the rest of physics and from what we are doing now. The principle of least action is universal. It applies in classical mechanics, quantum mechanics, special and general relativity, Maxwell electrodynamics, quantum field theory, etc. The hamiltonian, derived from the lagrangian, and the energy are always the same thing, etc.

We are going to use some conservation laws, plus the basic principle of stability which says the following : equilibrium happens at stationary points of potential energy. And we are going to use the mechanics of an object orbiting, but not an object in Newtonian mechanics, an object moving with the Schwarzschild metric given by equation (2).

We talked about it in the last lesson. We shall review it quickly. And then we will go into the calculations. We start with the action. Whenever you are working out equations of motion for anything, it is almost always easiest to start with the action principle. From the action principle derive a lagrangian, and from the lagrangian derive the Euler-Lagrange equations of motion. Or, from the action, use some other trick of the same kind.

We are going to use conservation laws. So we start with the metric, which we repeat below

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{2MG}{r}}\right) dr^2 - r^2 d\Omega^2$$

And the action of a moving particle [2] along a trajectory is

$$A = -m \int d\tau \tag{4}$$

Remember that an action has units of energy multiplied by time. So in order for the units to be correct in equation (4), there has got to be a mass in front. There is also an implicit $c^2 = 1$, because we know that mc^2 is an energy. In this chapter we don't need to consider explicitely the speed of light, so we work with $c = 1$.

And the integral in equation (4) is taken along the trajectory of the particle.

The proper time increment $d\tau$ is the square root of the right hand side of the metric given above, which defines $d\tau^2$. Let's write the action more explicitely. But in order not write over and over the coefficient $(1 - 2MG/r)$, let's call it $\mathcal{F}$, and let's call its inverse $\mathcal{G}$.

With these notations, the action can be written

$$A = -m \int \sqrt{\mathcal{F} dt^2 - \mathcal{G} dr^2 - r^2 d\Omega^2} \tag{5}$$

2. We begin the calculations with a particle of a certain mass. Then we shall turn to photons.

We are considering a particle circling around the black hole, so in the coordinates $(t,\ r,\ \theta,\ \phi)$, we can take one of the angles to be fixed and the other to be the angular parameter of the orbit. Suppose ϕ is fixed, then $d\phi = 0$, and, from equation (3), we see that $d\Omega^2 = d\theta^2$.

By now we should be familiar with the next step : we divide by dt^2 inside the square root, and multiply by dt outside the square root. Equation (5) becomes

$$A = -m \int \sqrt{\mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2}\ \ dt \tag{6}$$

For a circular orbit, $\dot{r} = 0$. And the formula will be a little bit simpler, but we don't want to quite go there yet.

What do we read off the lagrangian ? The lagrangian is the thing under the integral sign, when $-m$ has been brought in as well. Let's call it $\mathcal{L}$.

$$\mathcal{L} = -m\ \sqrt{\mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2} \tag{7}$$

There are two conservations laws that we want to use for this problem.

1. *Conservation of energy* : when the lagrangian of a system doesn't explicitely depend on t, the energy is conserved.
2. *Conservation of angular momentum* : the angular momentum of an orbiting particle is the conjugate momentum to θ. When the lagrangian is invariant under rotation, it is also conserved [3].

3. The angular momentum is the analog for rotating objects of the

Remember the definition of conjugate momentum in lagrangian mechanics : if we have a coordinate called q – also called a *degree of freedom* – the momentum associated with q is equal to the derivative of the lagrangian with respect to $\dot{q}$. We have seen and used that over and over in volume 1. Here it is again. Our degree of freedom is the angle θ. And let's call L its conjugate momentum. It is the angular momentum.

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \tag{8}$$

Using the expression for $\mathcal{L}$ given by equation (7), and the chain rule of differentiation, we first write the derivative of the square root of the intermediate variable

$$g = \mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2$$

then multiply it by the derivative of g with respect to $\dot{\theta}$, carrying along the factor $-m$. Two factors 2 and two minus signs cancel. We get

$$L = \frac{mr^2\dot{\theta}}{\sqrt{g}} \tag{9}$$

where $\sqrt{g}$ stands for $\sqrt{\mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2}$.

Since the lagrangian $\mathcal{L}$ doesn't depent on θ – only on $\dot{\theta}$ –, the angular momentum of the particle, given by equation (9), is constant. It is also proportional to the mass m of the particle.

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linear momentum – simply called *momentum* – for objects moving in a straight line. And remember that, if $\mathcal{L}$ is invariant under translation, the momentum of the object, or system of particles, is conserved.

We are going to express L, using the Greek uppecase letter lambda, as follows

$$L = m\Lambda \tag{10}$$

In other words, Λ is everything on the right hand side of equation (9) with the exception of the mass.

Equivalently, we can write

$$\Lambda = \frac{L}{m} = \frac{r^2\dot{\theta}}{\sqrt{\mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2}} \tag{11}$$

This quantity Λ is sometimes called the *reduced angular momentum*. It is the angular momentum per unit of mass. It is a conserved quantity because L is conserved, and m is a constant. And we see that Λ is a function of r, $\dot{r}$ and $\dot{\theta}$.

For a circular orbit it simplifies a little bit because $\dot{r}$ is equal to 0. But let's not do the circular orbit yet.

L is the momentum associated with θ. There is another momentum, P_r, associated with r – the other degree of freedom of the system. To find P_r we also have to differentiate the lagrangian, this time with respect to $\dot{r}$:

$$P_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{m\mathcal{G}\dot{r}}{\sqrt{g}} \tag{12}$$

where, as before, $\sqrt{g}$ stands for $\sqrt{\mathcal{F}(r) - \mathcal{G}(r)\dot{r}^2 - r^2\dot{\theta}^2}$.

Unlike the angular momentum L, the radial momentum P_r is not conserved. The lagrangian depends on r. As a consequence there is a radial potential. Therefore there are radial forces, that we could calculate from the Euler-Lagrange equations. There are accelerations along the radial axis.

If there are accelerations along the radial axis P_r cannot be conserved. But what is conserved is the energy. So we are going to write the general expression for the energy.

It is the good old hamiltonian. Remember the expression of the hamiltonian (see volume 1, chapter 8, equation (4)). If we have N generalized coordinates q_i, with i running from 1 to N, and the p_i's are the conjugate momenta, and $\mathcal{L}$ is the lagrangian, then the hamiltonian H is

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} \tag{13}$$

The point is that there is always an energy. It is always given by a certain construction – equation (13) – that *defines* it. And the energy is conserved so long as the lagrangian $\mathcal{L}$ does not explicitly depend on time. So here the energy is conserved.

In our case, there are only two coordinates, r and θ. From equation (13), we can calculate H. We get

$$H = \frac{\mathcal{F}(r)m}{\sqrt{g}} \tag{14}$$

We won't need P_r any further. We only needed it to calculate the energy. And H, in equation (14), is the energy. Henceforth we shall denote it E.

$$E = H$$

And it is conserved. It does not change with time.

Let's reexpress the energy with equation (14), but now we shall write explicitely the quantity that we denoted g, and we are going to use the fact that we are on a circular orbit. On a circular orbit, the radius doesn't change. So $\dot{r} = 0$. And we can write

$$E = \frac{\mathcal{F}(r)m}{\sqrt{\mathcal{F}(r) - r^2\dot{\theta}^2}} \tag{15}$$

This is partly kinetic energy due to the motion of θ. There is no kinetic energy due to the motion of r, because r is not changing on a circular orbit.

The energy E given by equation (15) is a combination of potential energy, because it depends on r, and kinetic energy because it also depends on the angular velocity $\dot{\theta}$. And it is conserved.

What about the reduced angular momentum ? We got it in equation (11). It has the same denominator as the energy, and now can be written.

$$\Lambda = \frac{r^2\dot{\theta}}{\sqrt{\mathcal{F}(r) - r^2\dot{\theta}^2}} \tag{16}$$

It is also conserved.

Now what do we do with these two conserved quantities, given by equations (15) and (16) ?

First of all, we solve equation (16) for $\dot{\theta}$. This will give us the angular velocity as a function of the reduced angular momentum. It is very easy to do and is left to the reader.

Then we plug it back into equation (15) for the energy. The result will be an expression of the enery E as a function of r and the reduced angular momentum. We will write in a second what we get.

Remark : physics always consists of this alternation between principles – having fun figuring out the principles, and what they tell us to do –, and then the boring work of doing a little calculations. And then back to the principles again.

Let's shortcut the boring calculations part. Here is what we get for the energy

$$E = m \, \frac{\sqrt{\mathcal{F}(r)} \, \sqrt{r^2 \Lambda^2 + r^4}}{r^2} \tag{17}$$

This does not depend on $\dot{r}$. It depends only on r and Λ.

What we are interesting in is the value of r which corresponds to a position of equilibrium of the energy. The equilibrium position depends on the angular momentum, or its reduced version Λ.

In ordinary Newtonian physics, for a given energy, is the angular momentum bigger or smaller the further away we orbit from the center ? The answer is not totally obvious. It is left to the reader to work out whether it is bigger or smaller as r gets bigger.

Now, we want to go to the limit of a photon. We are interested in the bizarre question of whether a photon can orbit in the geometry. Can a photon get close enough to the black hole that even the speed of light is not enough to eject it from a circular orbit around the black hole ?

So we are going to look for a circular orbit. And the rule is to minimize the energy as a function of r for fixed angular momentum. But now there is something going on : photons have zero mass. So we got a bit of a funny problem here.

Since photons have zero mass, it looks from equation (17) that they have zero energy. But they don't have zero energy ! A moving photon, for instance a high energy photon, doesn't have zero energy. Photons do have energy. So something had better get big to cancel the fact that m goes to zero.

Photons also have momentum and angular momentum. A photon moving around in orbit, or even if it is just going past the star does have an angular momentum. Remember, however, that the angular momentum L satisfies the equation

$$L = m\ \Lambda \tag{18}$$

What happens as m goes to zero ? The thing that you hold fixed is the angular momentum L. Necessarily Λ becomes infinite.

In other words, the right way to take the limit of a photon, that is, to study the trajectory of a photon of fixed energy and fixed angular momentum, is to take the limit as the mass m goes to zero, and the reduced angular momentum Λ goes to infinity, in such a way that the product of m times Λ stays fixed.

What happens in equation (17) when m goes to zero and Λ gets very big, because $\Lambda = L/m$ and L is a given ? As

Λ gets very big, the term $r^2\Lambda^2$ in the second square root eventually makes r^4 negligible. The square root of $r^2\Lambda^2$ is just $r\Lambda$. It is multiplied by m. So, taking into account that $m\Lambda$ is L, equation (17) becomes

$$E = L\,\frac{\sqrt{\mathcal{F}(r)}}{r} \tag{19}$$

This shows that, for a given angular momentum L, which is just a multiplicative factor, the energy of the photon is a function only of r. You can think of this E as a kind of potential energy.

The radial velocity $\dot{r}$ has been eliminated out, because on a circular orbit there is no radial velocity. The angular velocity $\dot{\theta}$ has been eliminated by expressing it in terms of the angular momentum L, which is a given. That is why E depends only on r.

Let's study the energy E of the photon as a function of r, in particular its point of equilibrium.

Photon sphere

Where is the equilibrium of E in terms of r? To find the value of r, all we have to do is differentiate the left hand side of equation (19) with respect to r and set the derivative equal to zero. At that value of r, the function $L\sqrt{\mathcal{F}(r)}/r$ will either have a maximum or minimum value.

Remember that the Schwarzschild factor, that we decided to denote $\mathcal{F}(r)$ to make the expressions lighter, is

$$\mathcal{F}(r) = 1 - \frac{2MG}{r}$$

Let's plot the function given by equation (19) to see what it looks like, and identify the point where it is stationary, figure 7.

Figure 7 : E as a function of r.

When r becomes very big, E is asymptotically equal to L/r. And when r gets close to the horizon, $2MG$, E gets close to zero. In between E is positive, and there is a point where its derivative is zero. Elementary calculations left to the reader show that the point where E is stationary is at

$$r = 3MG \tag{20}$$

It does not depend on the angular momentum. The interesting fact is that where E is stationary, it is at a maximum.

If you have a potential energy that has a maximum, does that correspond to an equilibrium position ? Answer : yes, but it is an unstable equilibrium. It is like placing a marble ball on top of a smooth metal sphere : if you place it absolutely precisely, the marble will stay there. But the slightest misplacement off the top, or the slightest tap on the marble, and the marble will roll off.

The smaller the tap you give it, however, or the smaller the imprecision in putting it right up at the top, the longer it will last at or near the top before eventually rolling down.

Here we are talking not about a marble and sphere, but about a circular orbit of a massless particle around a black hole. What we found is that right at $r = 3MG$ a photon can orbit a black hole. It is not at the Schwarzschild radius. It is at one and a half times the Schwarzschild radius. The sphere at that radius is called the *photosphere*, or the *photon sphere*. The obvious reason is that it is a sphere on which a photon can orbit.

You might expect that if you had a black hole, and you looked at that distance you would find all kinds of photons orbiting around it. After all any photon that gets started in that orbit, will stay in that orbit.

But of course if the photon starts slightly away from the photon sphere, it will whip around and come out – not necessarily in the opposite direction. The closer you get to the photon sphere the more angle the photon will sweep before it eventually goes away. A photon coming in very close can circle around many times before escaping, figure 8.

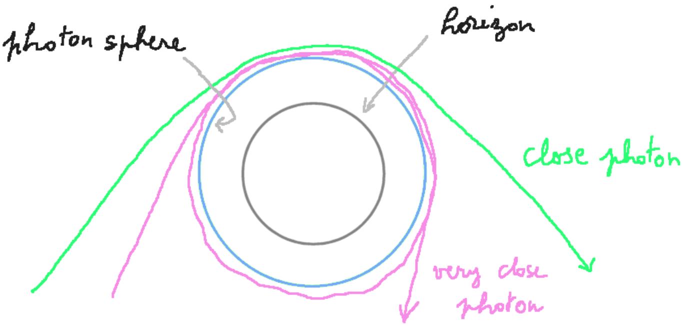

Figure 8 : Photons passing slightly outside the photon sphere.

Photons whipping around, or temporarily orbiting the black hole before escaping, correspond to a closest distance to the black hole slightly off to the right of the point $3MG$ in figure 7, and then slowly going outward.

What happens, on the other hand, if a photon is on the left of $3MG$ in figure 7 ? Answer : it will get pulled in. It can't withstand the gravitation if it is just slightly inward the photon sphere. It might orbit a few times, as above, but it will spiral into the horizon and then of course eventually spiral into the singularity.

This was an example of calculations with the Schwarzschild metric. And we showed the surprising fact that massless particles can have an unstable equilibrium orbit at radius $r = 3MG$.

We recommend the reader to compare the calculations we did with the corresponding Newtonian calculations. He or she will see all the pieces are the same. But the outcome is quite different.

In Newtonian physics, light-rays don't orbit anything. In inertial frames of reference, they move in straight lines whatever the field they go through. In inertial frames of reference, massive particles, plunged in a central gravitational field, move along conics. And for certain configurations of E and L massive particles can follow a circular orbit around a force center at any distance, including at distance $r = 3MG$ or smaller.

Whereas, in space-time with the Schwarzschild metric, particles below the photosphere, whether massive or massless, will inescapably get sucked in.

The phenomenon of photons temporarily orbiting in the photon sphere before escaping again has interesting consequences. Let's look at a black hole from a distance. The black hole itself emits no light, but light-rays coming from outer space behind the black hole, relative to you, display strange patterns.

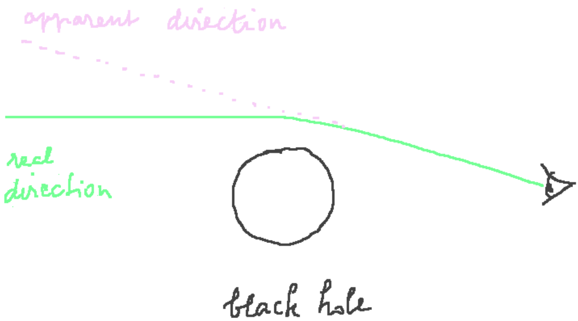

Figure 9 : Distant light emitter apparently displaced.

For instance a light-ray arrives from a light emitter, say, a distant star, in outer space behind the black hole. It bends a little bit, and then hits your eye, figure 9. So instead of seeing the light-ray coming from its real direction in the backdrop, you see it apparently coming from another direction.

It is worse than that. A light-ray coming from the same distant source, may fly by the black hole near another point, for instance not above but below the black hole in figure 9. Then it will be bent differently and appear to be coming from another direction than the first one. Therefore the light emitter behind the black hole will appear to your eye as a ring around the black hole. A black hole is a bad place for your ennemy to hide behind from you.

What you see is even worse. Other points in the backdrop may emit light-rays that will come near the photosphere, orbit for a while and happen to leave in a direction that will hit your eye, figure 10.

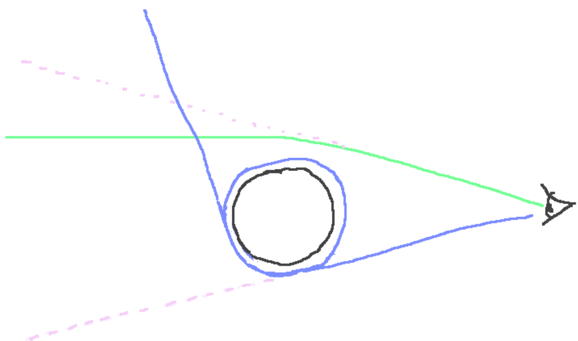

Figure 10 : Another light emitter apparently displaced.

Light emitters don't have to be far away to display this phenomenon. Light emitting points near the horizon of the black hole, anywhere on the sphere, can emit light-rays that will come out of the photosphere. Some can end up in your eye.

In short, what you will see from the backdrop as well as from the vicinity of the black hole will form a complicated pattern in your eye due to light-rays trajectories modified in various ways by the black hole. That makes looking at a black hole a dizzyfying experience.

Notice that the ring created by a source behind the black hole [4], called Einstein's ring, is a physically observed phenomenon. The first complete one was observed by the Hubble space telescope in 1998.

Exercise 1 : Explain why a light-ray emitted from inside the photon sphere can escape, but a light-ray cannot enter the photon sphere and come out again.

4. Black holes are convenient massive bodies to observe light deviation, because they don't emit light on their own. Therefore, unlike with the Sun for instance, we don't have to wait for an eclipse to study light-rays flying in their vicinity. Cf the topic of gravitational lens.

Hyperbolic coordinates revisited

Now we want to understand better what is happening at the event horizon of the black hole, figure 11. Among other things, something curious happens in the equation of the metric.

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{2MG}{r}}\right) dr^2 - r^2 d\Omega^2$$

The sign of the coefficient in front of dt^2 changes. At the same time the sign of the coefficient in front of dr^2 changes too – thus preserving the signature of the metric.

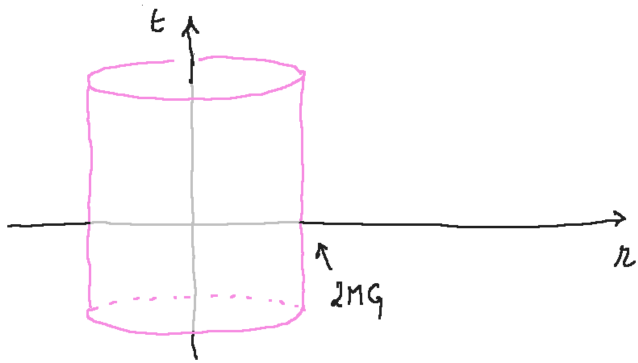

Figure 11 : Black hole horizon.

We know that clocks "slow down", in the sense that when r gets close to $2MG$, for a given dt the corresponding $d\tau$ is smaller and smaller. We saw that a stick falling in vertically contracts, etc. But I also told you that in some other sense, nothing is happening : someone falling in will not experience anything special while crossing the horizon.

Many things that happen in gravity can be understood by first understanding them for accelerated coordinate frames. That is the equivalence principle.

Gravity is not exactly the same as a uniformly accelerated coordinate system of course, but it is often a good idea, when studying a question, to first look at how it manifests in a uniformly accelerated coordinate system. So let's go back to that problem.

Remember that studying a uniformly accelerated reference frame in relativity is a little more complicated than it is in Newtonian physics. It is best studied in a coordinate system which is the analog of polar coordinates – but hyperbolic polar coordinates. Let's do a quick review.

We are in a flat space-time now. It is not a black hole, it is not the gravitational field of a massive body, it is plain old flat space-time.

Any point P has standard Minkowski coordinates, which we denote $(T,\ X)$, and hyperbolic polar coordinates, which we denote $(\omega,\ \rho)$. There may be more spatial coordinates Y and Z, but we are not interested in them, because they won't play a role in the subsequent analysis.

We have called coordinate ω the *hyperbolic angle*, because it is the analog of the ordinary angle in polar coordinates. However, instead of varying from 0 to 2π, it varies from $-\infty$ to $+\infty$. It is a kind of time. Coordinate ρ is sometimes called the *hyperbolic radius*. A point in figure 12 can be located with its T and X coordinates, and also with its ω and ρ coordinates.

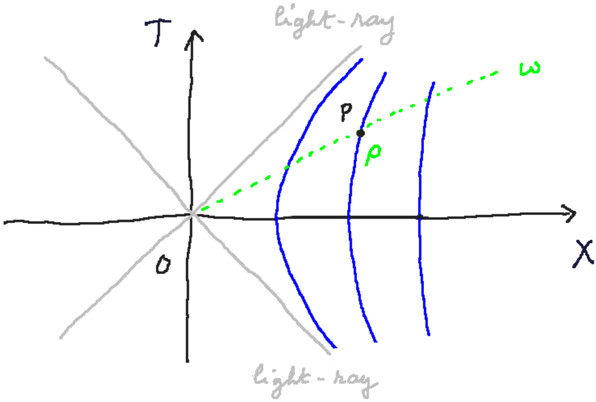

Figure 12 : Flat space-time with hyperbolic polar coordinates.

Remember the following facts in figure 12 :

- points on a horizontal line have the same T
- points on a vertical line have the same X
- points on the same dotted line have the same ω
- points on the same hyperbola have the same ρ

Finally the algebraic correspondence between the Minkowski coordinates and the hyperbolic polar coordinates of P is

$$\begin{aligned} X &= \rho \cosh \omega \\ T &= \rho \sinh \omega \end{aligned} \tag{21}$$

What about $X^2 - T^2$? It is $\rho^2(\cosh^2 \omega - \sinh^2 \omega)$ which is equal to ρ^2. So on each hyperbola in figure 12, $X^2 - T^2$ is constant when ω varies.

In chapter 4 we studied such a transformation. It defines a *uniformly accelerated reference frame in special relativity.* Let's recall briefly the reasoning :

1. In Newtonian physics, the concept of a uniformly accelerated collection of points, fixed relative to each other, is intuitive and clear.
2. In Newtonian physics, points rotating around a circle, all fixed relative to each other, are all submitted to the same acceleration toward the center.
3. Finally, in special relativity, points with fixed ρ's and moving with the same ω, are said to be uniformly accelerated. In this latter case, it is a definition.

The metric of ordinary flat space-time, expressed in the hyperbolic polar coordinates, becomes

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2 \tag{22}$$

That is the analog of $r^2 d\theta^2 + dr^2$ in ordinary polar coordinates on the Euclidean plane, that is in ordinary geometry. And equation (22) is the same in Minkowski space-time.

Now, in order to study conveniently what is going on in the vicinity of the event horizon of a black hole, we are going to change coordinates. We don't change coordinates for the pleasure, but to make life simpler. The new coordinates will be denoted ξ and t, and defined as follows

$$\begin{aligned} \xi &= \frac{\rho^2}{4} \\ t &= 2\omega \end{aligned} \tag{23}$$

And we rewrite the metric. Elementary algebraic manipulations on equation (22) yield

$$d\tau^2 = \xi dt^2 - \frac{d\xi^2}{\xi} \tag{24}$$

We notice a similarity with the Schwarzschild metric. In the Schwarzschild metric, equation (2), which we reproduced at the beginning of this section, we have a coefficient in front of dt^2, and we have the inverse of the same coefficient in front of dr^2. If we think of ξ as r, equation (24) bears some family resemblance.

Notice another thing : $(1 - 2MG/r)$ and its inverse change sign when r gets in too close to the singulariry. Somehow analogously, ξ changes sign simply when it becomes negative, and then so does $1/\xi$. Does ξ becoming negative means something ? Yes. We will see that ξ becoming negative does mean something.

So there is a similarity. But the basic similarity is a coefficient and another coefficient, which are inverses of each other, and which, we will see, meaningfully change sign as ξ become negative.

In what follows, we are going to use the flat metric of equation (24). But let's first turn again to the Schwarzschild metric of the black hole.

On the black hole, we want to zoom in on the place where $r = 2MG$. This is another of saying that we shall study the Schwarzschild metric of equation (2) in the region where r is almost equal to $2MG$. We use a microscope to look at points near the horizon.

The first coefficient in the Schwarzschild metric is $(1 - 2MG/r)$, which can be rewritten

$$\frac{r - 2MG}{r}$$

Since we won't let r change very much near $2MG$, to the first order approximation (in $\epsilon = r - 2MG$) it is equal to

$$\frac{r - 2MG}{2MG}$$

This is like in Newton's equation : if you are near the surface of the Earth, for many purposes you can set r equal to the radius of the Earth.

Omitting for the moment the term $r^2 d\Omega^2$ which only concerns changes in the spherical angles, the Schwarzschild metric becomes

$$d\tau^2 = \left(\frac{r - 2MG}{2MG}\right) dt^2 - \left(\frac{2MG}{r - 2MG}\right) dr^2 \tag{25}$$

In order to simplify the expression, we shall set the mass of the black hole to a specific number, so that $2MG$ equals 1. We can always do that without loss of generality. It is just a choice of units.

What are the units of $2MG$? Units of length, because $2MG$ is the Schwarzschild radius. For a particular black hole we can choose units of length where the Schwarzschild radius is just equal to one. We can't do it for all black holes simultaneously but if we are interested in a particular black hole we can do that. That makes formula (25) much simpler. It is now

$$d\tau^2 = (r-1)dt^2 - \left(\frac{1}{r-1}\right) dr^2 \tag{26}$$

The next step is to redefine $r-1$ and call it ξ. It is just a change of variables. Since with our preceding change of units the horizon had coordinate $r=1$, we are now translating coordinates so that the horizon has coordinate $\xi = 0$. And ξ measures the deviation from the horizon. Equation (26) becomes

$$d\tau^2 = \xi dt^2 - \frac{1}{\xi} dr^2 \tag{27}$$

Notice that dr and $d\xi$ are the same thing since $\xi = r-1$. So after all these various changes (local analysis at the horizon, change of units of mass of the black hole, and change of variable near the horizon) the Schwarzschild metric is now

$$d\tau^2 = \xi dt^2 - \frac{d\xi^2}{\xi} \tag{28}$$

This equation (28) is exactly the same as equation (24)! This is telling us is that the vicinity of the horizon is in some mysterious way just *flat space-time*, or approximately flat space-time. Nothing dramatic is happening there. No large curvature, only more or less flat space-time. This may be surprising because of what we saw earlier – the slowing down of clocks, the contraction of lengths, etc. – but these were just coordinate-related phenomena. With an appropriate change of coordinate, the space-time of the black hole displays no special features at the horizon. It is locally more or less flat like everywhere else (except at $r=0$).

Of course, the space-time near the horizon is flat in a uniformly accelerated coordinate system – just like on the surface of the Earth inside a free-falling elevator there is no gravity. Uniform gravity or uniform acceleration don't make the space non-flat. The essential point in the above analysis is that nothing special happens at the black hole horizon.

Why would we be interested in a coordinate system, for a gravitating object, which is mimicking uniform acceleration ? Well, standing where I am, I am experiencing an acceleration of 10 meters per second per second. I feel exactly the same things that I would if I were in free space and the floor under me was being accelerated upward with an acceleration of 10 meters per second per second.

Similarly, in studying the gravitating black hole from the perspective of somebody standing still, being supported let's say above the horizon of the black hole or wherever he or she is, you are effectively doing physics in a uniformly accelerated reference frame in space-time.

What does physics feel like there ? What does it do there ? It does whatever the uniformly accelerated reference frame does. And that is exactly what we experience near the horizon of a black hole : we experience whatever is effectively created by a uniformly accelerated coordinate system – just like you, in Newtonian physics, sitting in your chair. But the space-time itself is flat.

Interchange of space and time directions at the horizon

Now let's talk about the interchange of space to time and time to space, that we have already alluded to, when we cross the black hole horizon.

We shall now make full use of the fact we just established, that in the vicinity of the black hole horizon the space-time gravitational field is equivalent to that of a uniformly accelerated reference frame.

We saw that we defined ξ as

$$\xi = r - 1$$

or, if we put back M and G, ξ would be proportional to $r - 2MG$.

ξ can change sign. How ? It changes sign by going from outside the horizon to inside the horizon. It makes sense – at least *in the black hole context* – to say that ξ can change sign [5].

This requires some clarifications because in a uniformly accelerated reference frame in space-time, from equations (23), we also have

$$\xi = \frac{\rho^2}{4} \tag{29}$$

5. Notice that, even though in the Schwarzschild metric, space-time is everywhere locally flat – except at $r = 0$ –, there is still a special role played by $r = 2MG$, coming from the definition of the metric in equation (2).

How can ρ^2 change sign ? ρ is a real number and ρ^2 is always positive. But that is not the right way to look at ρ^2.

The right way to look at ρ^2 is to look at the right hand side of

$$X^2 - T^2 = \rho^2 \tag{30}$$

For a fixed right hand side, this is the equation of a hyperbola. If ρ^2 is positive, it is a hyperbola in the first quadrant and third quadrant. If ρ^2 is negative, it is a hyperbola in the second and fourth quadrant, see figure 13. Let's forget about the left and lower quadrants, and concentrate on the right and upper quadrants.

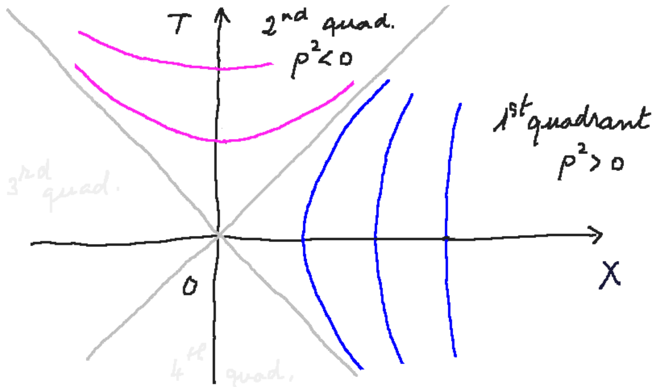

Figure 13 : Hyperbolas with positive ρ^2 and with negative ρ^2. The point O is not the center of the black hole, but a point on its horizon.

$\rho 2$ being negative simply corresponds to points in the second or upper quadrant. So in going from ξ positive to ξ negative, we are simply passing from the right quadrant to the upper quadrant.

Imagine that we follow the space coordinate ξ till we get to $\xi = 0$. Then where would we go from there?

In other words, we are somewhere on the horizontal axis. The hyperbolic angle ω is, for instance, equal to zero. And we are following our way with ξ decreasing, while $\omega = 0$, toward the origin. At the origin O – which, remember, is not the center of the black hole, but the point where we cross the horizon (with θ and ϕ fixed) –, ξ changes sign. Then where do we go, if we keep moving with ξ now varying in the negative numbers, and ω still zero? We don't go left into the third quadrant, we now go upward into the second quadrant.

On the OX axis, in figure 13, $\omega = 0$ and $\xi > 0$. And on the OT axis, $\omega = 0$ and $\xi < 0$.

What happens is that a space-like coordinate, namely ξ, jumps to become a time-like coordinate. Indeed, when ξ is positive, a variation in ξ entails an interval displacement of the space-like type. Then, when ξ is negative, a variation in ξ entails an interval displacement of the time-like type.

Let's now turn to the coordinate ω. In the first quadrant, ω – also called the hyperbolic angle – is a time-like coordinate. When it varies alone, we move on a time-like interval. What happens to it after we passed the origin O? Now when we are in the second quadrant, and let ω vary, with ξ fixed, we are moving along a hyperbola of the second quadrant, see figure 13, that is along a space-like interval. In summary, when we pass O, in other words when we cross the horizon, the space and time directions in the coordinate system $(\xi,\ \omega)$ are interchanged.

Let's not be mistaken : ξ remains ξ, and ω remains ω. But the former, which was space-like, becomes time-like. And the latter, which was time-like, becomes space-like.

All these considerations concern *coordinates* of events in space-time. Nothing physically meaningful happens at the events or, more accurately, to the particles going from event to event, following trajectories in space-time [6] – with the exception of the singularity at the center of the black hole, which we shall study in the next section.

ξ and ω are just funny coordinates where ξ goes from being a space coordinate to being a time coordinate, and conversely ω goes from being a time coordinate to a space one.

However, remember that all this is just a flat space. There is nothing going on in the space-time region around the horizon. Somebody who jumps from the right quadrant to the upper quadrant sees nothing special. There is nothing happening. It is just peculiar coordinates that have some funny behavior as you go from one quadrant to the other.

The same is true of the Schwarzschild coordinates. ξ is $r-1$ or $(r-2MG)/r$. When $r-2MG$ changes sign, you are doing

6. It is important to understand that time is just one of the coordinates to locate events in space-time. We have freedom in selecting what is our time coordinate, *provided that when it varies alone we move on a time-like interval.* We have used in turn T in the standard Minkowski diagram, ω in equations (21), t in equations (23), etc. And a trajectory is by definition a collection of events indexed by a time coordinate.

exactly the same thing as you would going from the right quadrant to the upper quadrant in figure 13.

Black hole singularity

Let's go on and think about the black hole itself now.

The coordinate ξ is simply $r-1$. Decreasing r until we got to 1 consisted in reaching and then crossing the event horizon. This also consisted in reaching and crossing the origin in figure 13. Then continuing to let r decrease toward 0, we were now moving up along the vertical axis. That is the nature of these coordinates, and is not the most important point from a physical point of view.

Eventually we will reach $r=0$. And that is what we want to study now.

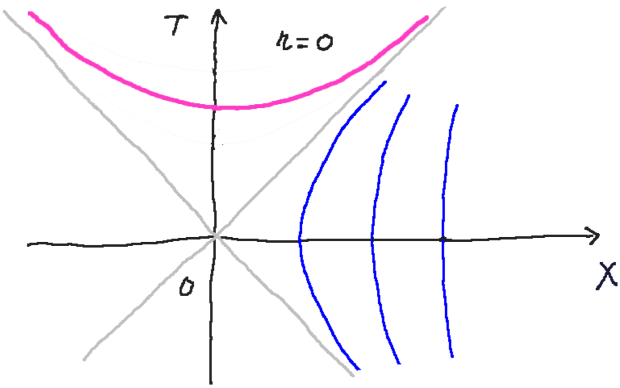

Figure 14 : $r=0$. It is not a vertical straight line, but a time-like curve.

At $r = 0$ something nasty happens. Where is $r = 0$? It is in the upper quadrant on the hyperbola shown in figure 14.

At $r = 0$ we are behind the horizon, inside the sphere of radius $r = 2MG$. The coefficient $(1 - 2MG/r)$ has changed sign and went to $-\infty$.

And lo and behold : the singularity at $r = 0$ is not a place, it is a time. More accurately, it is not a single place but many different places all on a time-like curve. The surface $r = 0$ is not what we would normally think of. Normally a place corresponds, in the Minkowski diagram, to a vertical line. But this is not the case of the singularity at $r = 0$. In this case it is the hyperbola in the upper quadrant in fig 14. And it is more time-like than space-like.

This is difficult to figure out and at first confusing of course. But this is the way the black hole really is. You will get used to it, if you think about it and play with it.

We cannot avoid the singularity when we got past the horizon. Why ? Because we cannot avoid the future. There is no way to escape from the future the way we can avoid a place. Think of a place in front of you : you can avoid it, you can go around it. What you cannot do is avoid the future, any future time. Once we are past the origin in fig 14, once we are inside the horizon, that is in the upper quadrant, no matter what we do we will run into the hyperbola $r = 0$.

The singularity is not avoidable the way it would be in Newtonian physics. In Newtonian physics the center of the coordinates is a place. We can go around a place, but we cannot go around a time. So that is the nature of the black hole singularity.

What about the horizon ? The horizon is the point O in figure 14. But we are going to think about it a little bit differently eventually. We are going to think about the whole straight line at 45° as being the horizon.

Figure 14 explains something to us. It allows us to understand much better the idea that it takes an infinite amount of time for a thing to fall through the horizon – at least an infinite amount of *coordinate time*. But still, it falls through in its own time keeping. And with that time it falls through in a finite time. Let's redraw the figure and see if we can understand that.

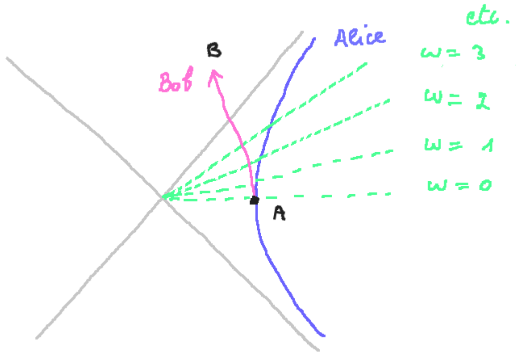

Figure 15 : Alice stays outside the black hole horizon and Bob falls in.

Consider the two famous protagonists : Alice and Bob. Alice, one way or another, stays outside the horizon of the black hole *at a fixed distance ρ from the horizon.* And she pushes Bob or at least observes him fall into the black hole.

The origin, in figure 15, is the point on the horizon below her, as in figure 14. The straight lines from the origin are time slices of constant ω. Remember that ω is like time. The relationships between standard Minkowski coordinates and hyperbolic coordinates are given by these equations [7]

$$X = \rho \cosh \omega$$
$$T = \rho \sinh \omega$$

So Alice's trajectory is the hyperbola in fig 15. Even though she doesn't move, it is not a vertical straight line because of the Schwarzschild metric which is a gravitational field. And we saw that, at any point except the center of the black hole, it is everywhere locally equivalent to a uniformly accelerated frame, even in the vicinity of the horizon. So a motionless point above the horizon follows a hyperbola.

Alice's ρ doesn't change, and the time, measured by ω, ticks as she passes each radial line in the picture. Remember that ω is proportional to her proper time, see equation (22). ω asymptotically goes to infinity when the radial lines becomes closer and closer to the 45° line. Alice says that the surface with 45° slope corresponds to infinite time. Notice that the coordinate time T – the time of the physicist observer of the whole thing if you like – becomes infinite if and only if ω becomes infinite.

If we consider now Bob unfortunate trajectory, it is not until an infinite amount of Alice's proper time has elapsed,

7. We have only translated the horizontal axis so that the origin of the diagram is not at the center of the black hole, but on its horizon.

that Bob has gone through the horizon. In other words she never sees Bob go through the horizon. According to her reckoning, it takes an infinite amount of ω before Bob falls through the horizon. That translates to an infinite amount of T.

On the other hand, just looking at the diagram in fig 15, we can see immediately that the amount of proper time it takes Bob to go from point A to point B is finite. In other words Bob goes through the horizon in a finite time. The surface $X = T$ *is* the horizon. We will come back to it.

It is only a funny coordinate property which leads to Alice's reckoning that Bob takes an infinite amount of time to fall through the horizon, while Bob says : no, I watched it on my clock, it only takes a finite amount of time. The reason is the discrepancy between proper time and coordinate time. And Alice uses her proper time, which is linked to coordinate time by $T = \rho \sinh \omega$, while Bob uses his proper time.

We can figure out how Alice "sees" Bob. It requires somehow for Alice to look back in time, because when we speak of looking we refer to light arriving to our eyes, and light takes time to travel. But let's talk of "looking" in a different way : when we say Alice looks at Bob at time ω, we mean she figures out what does Bob do at this simultaneous time ω. In other words, where is Bob on the radial line of value ω. Well, we know from the picture where he is. So we see that Alice sees Bob slow down. Each heartbeat of Bob takes longer in Alice's time.

Now how does Bob see Alice ? We apply the same reasoning : by what Bob "sees" of Alice, we mean what happens to Alice at the same Bob's time, when he "looks back", that is going into the past coordinate time, but simultaneous for him. Those lines of simultaneous times for Bob are $-45°$ lines which link points on Bob's trajectory and Alice's trajectory, figure 16. So we see that Bob doesn't see anything special. He sees Alice before he plunges through the horizon. He sees Alice while he is at the horizon. And he even sees her afterwards without problem.

While on his trajectory, on which he does not notice anything particular happening when crossing $r = 2MG$, Bob doesn't see Alice shrink. He does see Alice accelerate away from him, that's true. But that is all. At any point, he sees Alice perfectly normally, except that she is accelerating away from him.

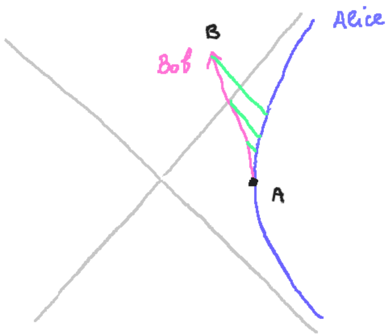

Figure 16 : How Bob sees Alice.

Notice that when Bob is at B, he can see Alice. Whereas Alice can never see Bob at point B. So it is a very asymmetric situation.

What happens when Bob hits the singularity? By that point the tidal forces will become so large that Bob will no longer be with us. Bob will have been destroyed by the time he gets to the center of the black hole field. That is why we don't think about trying to answer the question : what does Bob see when he gets to, say, a negative r. By the time he gets to negative r, he has experienced infinitely strong tidal forces and there is no more Bob.

Notice that in the upper quadrant, Bob will probably use r for time or something like r. Inside the horizon, where r varies from $r = 2MG$ to $r = 0$, Bob's clocks would have more to do with r than with t of equation (2). And t would be more like position. But, as I said, Bob doesn't feel any funny thing happening with space or time. His clock doesn't become a meter stick, and his meter stick doesn't becomes a clock. Nothing like that happens. He just sails through.

Yet the line at 45° in figure 16 is rather special. Let's see what can and cannot happen once something – a person, an object, a particle or even a photon – has crossed the horizon.

No escaping from a black hole

Anything that crossed the horizon can no longer escape.

Remember, in the coordinates that we are using light moves with a 45 degree angle. Therefore light cannot escape from the upper quadrant in figure 17. All it can do is eventually hit the singularity. And anything that is moving slower than the speed of light has a slope closer to the vertical, and will also hit the singularity. So anybody who finds himself or herself in the upper quadrant is doomed.

Somebody in the right quadrant has the possibility of escaping. Let's talk again about light-ray. If it is moving radially outward, it will escape. It will simply just keep going. If it is pointing radially inward it is of course also doomed. It will hit the singularity.

And there is everything in between. If a light-ray is pointing out of the page, depending on whether it is inside or beyond the photon radius, it will fall in or not.

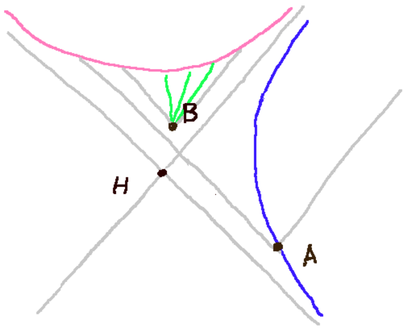

Figure 17 : Trajectories starting inside or outside the horizon.

Figure 17, and its variants fig 15 and 16, are the pictures you want to get used to. If you want to understand and be able to resolve funny paradoxes about who sees what in the black hole, you should go back to this picture.

Alice is outside the black hole at a fixed distance ρ, subjected to gravitation locally equivalent to uniform acceleration. Alice's time corresponds the dotted lines shown in fig 15. Up to the fixed factor ρ, her clocks record the hyperbolic angle ω, because Alice's proper time $d\tau$ is, with hyperbolic coordinates, $\rho d\omega$. See equation (22) which says

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2$$

Bob starts outside the black hole, and passes inside below $r = 2MG$. The trajectory he follows can be a free fall or any other trajectory. In fig 17, outside the black hole corresponds to the right quadrant, and inside the black hole corresponds to the upper quadrant. That is due to the peculiar Schwarzschild metric given by equation (2), whose first and second coefficients interchange signs at the horizon. Bob's clocks record Bob's proper time. Like for anyone else, it is $d\tau$ given by equation (2).

It is the strong mismatch between Alice's proper time and Bob's proper time which produces all the peculiarities showing up around a black hole.

A good understanding of black holes is a prerequisite to understanding general relativity. The reason is that black holes are the simplest, ideal form of massive bodies : their mass is concentrated at a point. It is the equivalent in relativity of point masses in Newtonian physics.

Point masses in Newtonian physics give rise to Newtonian gravitation, trajectories solved with Newton's equation, etc.

Their physics has a singularity at the central point itself, but nowhere else. In relativity the analog of point masses is black holes, and their metric. It is Schwarzschild metric. Black holes display of course a singularity at their center, but also strange phenomena at $r = 2MG$. However the peculiar phenomena happening at $r = 2MG$ only come *from the coordinates*, not from real physical strangeness.

The next two chapters will be devoted to deepening our knowledge of black holes.