Lesson 7 : Falling into a black hole

Notes from Prof. Susskind video lectures publicly available on YouTube

Introduction

The lesson is devoted to reviewing, getting acquainted with, and deepening what we began to learn in the previous lesson, namely what happens around a black hole, outside the horizon, at the horizon, and at the singularity.

We will go again over the change of coordinates and the corresponding diagram which facilitate understanding the physics at the horizon and in its vicinity.

We will study people and objects falling in, and possible and impossible communications between inside and outside.

The lesson ends up with miscellaneous notes from a question and answer session when it was given in front of students.

Schwarzschild metric, event horizon, singularity

Let's get back to the Schwarzschild metric, named after Karl Schwarzschild (1873 - 1916). In the early years of last century he was a professor at Göttingen, with among his colleagues Hilbert and Minkowski. I don't know much more about him because that was before my time. But we shall know his metric very well.

We shall simplify its equation a little bit by rescaling the coordinates. This won't change anything about the physics, but will make the equations simpler to work with.

Let's write it first as we learned it. But we put primes to the time and radius variables, so that after the change of variables, the new variables will be as usual $(t,\ r,\ \theta,\ \phi)$.

$$d\tau^2 = \left(1 - \frac{2MG}{r'}\right) dt'^2 - \left(\frac{1}{1 - \frac{2MG}{r'}}\right) dr'^2 - r'^2 d\Omega^2 \quad (1)$$

This is exactly the same equation as in the previous lessons, except with just a slight change of notation.

Let's give a new name to the Schwarzschild radius $2MG$:

$$R_S = 2MG \tag{2}$$

R_S reads "R Schwarzschild". Next, we rescale the radius. We *define* r as follows

$$r = \frac{r'}{R_S} \tag{3}$$

Normally radii have units of length. But when we divide a length by a length, we get something which is dimensionless. So our new r, which is not the same r that we used in the last lesson [1], is dimensionless.

We do the same thing with time. What are the units of time ? Time. But since in our equations we carry an implicit $c = 1$, which has units length over time, for us units of time and units of length are the same. Our new t is

$$t = \frac{t'}{R_S} \tag{4}$$

1. r' is the radius used in the last lesson.

You can think of R_S as both a distance and a time. It is the distance from the center of the black hole to the horizon. And it is also the time light takes to cover that distance.

Finally, the equation of the metric takes the following simpler form

$$d\tau^2 = \left[\left(1 - \frac{1}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{1}{r}}\right) dr^2 - r^2 d\Omega^2 \right] R_S^2 \quad (5)$$

Inside the brackets is the metric that would have written down in the first place if $2MG$ were equal to 1. What we find out from this exercise is that basically the metric of all Schwarzschild black holes is the same, except for an overall factor which is proportional to the square of the radius of the black hole. Moreover, everything inside the brackets is dimensionless. All of the dimensions are in R_S^2.

For most purposes in studying the geometry of the black hole, we can really just set R_S equal to one. It is like studying the metric of a sphere. The metric of all spheres is the same, with just a different radius. There are bigger spheres and smaller spheres, but, up to a factor which is their radius, they all have the same geometry. Analogously, all black holes are the same, up to their radius.

Now let's turn to a question we haven't asked ourselves for a while. What is the curvature of space-time near a black hole ? Remember the curvature tensor curvature tensor that we studied in lesson 3. It is the tool which tells us whether the space-time is flat or curved. It is analogous to tidal forces. It is the tendency for the geometry to distort objects. And in fact tidal forces are curvature.

The formula is

$$\mathcal{R}_{srn}^{\ \ \ \ t} = \partial_r \Gamma^t_{sn} - \partial_s \Gamma^t_{rn} + \Gamma^p_{sn}\Gamma^t_{pr} - \Gamma^p_{rn}\Gamma^t_{ps} \tag{6}$$

Our question is : how big are the components of the curvature tensor, let's say at the horizon ? The horizon corresponds to $r = 1$.

A preliminary question is : what are the units of curvature ? Remember that a Christoffel coefficient can be calculated. It is expressed in terms of the metric components as follows

$$\Gamma^t_{mn} = \frac{1}{2}\ g^{rt}\ \left[\ \partial_n g_{rm} + \partial_m g_{rn} - \partial_r g_{mn}\ \right] \tag{7}$$

We also remember that

$$dS^2 = g_{\mu\nu} dX^\mu dX^\nu \tag{8}$$

From equation (8), it follows that $g_{\mu\nu}$ is dimensionless. And so is its inverse $g^{\mu\nu}$. Then, since the Christoffel symbols, in equation (7), contain spatial derivatives of g, they have units one over length. And we conclude, from equation (6), that the curvature tensor components have units one over length squared. Using the standard notation with brackets for dimension, this is expressed by writing

$$[\mathcal{R}] = \frac{1}{m^2} \tag{9}$$

Let's come back to the question : what is the curvature at $r = 1$?

And let's use a nice argument resting only on dimensions to find the answer. Looking again at the metric, the only quantity on the right hand side of equation (5) which has units of length is R_S. Everything else is dimensionless. Therefore when we calculate the curvature – doing all sorts of algebraic manipulations on the dimensionless elements in equation (5) –, the only possibility for it to have units one over length squared is if the curvature is inversely proportional to R_S^2 itself, that is, to the Schwarzschild radius squared [2]. We write it

$$\mathcal{R}_{Horizon} \sim \frac{1}{R_S^2} \tag{10}$$

where $\sim$ stands for "proportional to", with a fixed proportionality ratio, the same for all black holes.

For a given black hole, if at $r = 1$ the curvature has a certain value, at $r = 2$ the curvature will be four times smaller. Conversely at radius $R_S/2$, the curvature will be four times bigger. As we approach the singularity, the curvature becomes infinite.

For different black holes, the bigger the black hole – i.e., the more massive it is – the smaller the curvature at its horizon. So the tidal forces associated with a large black hole are less severe than the tidal forces at a small black hole. A small black hole is a much nastier thing *near its horizon* than a very big black hole.

Now let's concentrate on the geometry created by the metric of equation (5). And let's throw away the R_S^2, in other

2. To convince yourself, think of what would happen if you multiplied R_S by any dimensionless constant k and changed the units before doing all the calculations.

words let's replace it by 1. We consider, if you like, a "unit black hole", that is, a black hole of radius 1.

Let's go back to something we already talked about in the last lesson. We have only more or less outlined it. Let's do it again. We want to use new coordinates. You may think that we keep changing coordinates but that will make things simpler. And it is what relativity is all about : it is changing coordinates.

We will introduce a new coordinate to replace r. Figure 1 below shows the r axis.

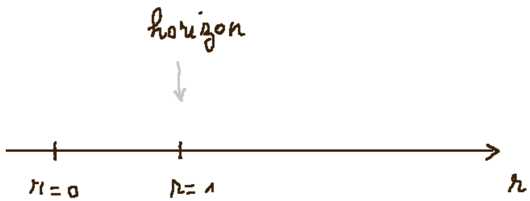

Figure 1 : Coordinate r.

We want to invent a new coordinate which is just *proper distance from the horizon.* How do we calculate the distance from the horizon ? We go back to the expression of proper time given in equation (5), which we rewrite below, with $R_S^2 = 1$,

$$d\tau^2 = \left(1 - \frac{1}{r}\right) dt^2 - \left(\frac{1}{1 - \frac{1}{r}}\right) dr^2 - r^2 d\Omega^2 \tag{11}$$

Remember that $d\tau^2$ is proper time squared, and dS^2 is proper distance squared. These are just minus each other. So proper distance is given by almost the same equation as equation (11)

$$dS^2 = -\left(1 - \frac{1}{r}\right) dt^2 + \left(\frac{1}{1 - \frac{1}{r}}\right) dr^2 + r^2 d\Omega^2 \tag{12}$$

Now let's move away from the horizon radially outward. Consider two points A and B at distance r and $r + dr$. What is the proper distance between these two points?

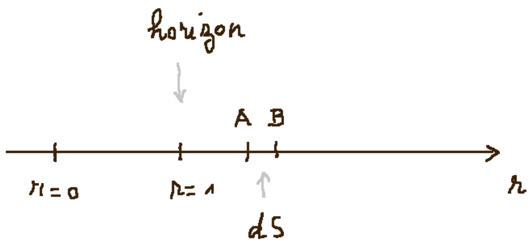

Figure 2 : Proper distance between A and B.

To find out we use equation (12). First of all we compare two points at the same time, so $dt = 0$. Secondly we move radially, so Ω doesn't change either. Therefore between A and B, the proper distance squared is

$$dS^2 = \frac{dr^2}{1 - \frac{1}{r}}$$

or

$$dS = \sqrt{\frac{r}{r-1}}\, dr \tag{13}$$

Let's give a name to the proper distance from the horizon. We call it ρ.

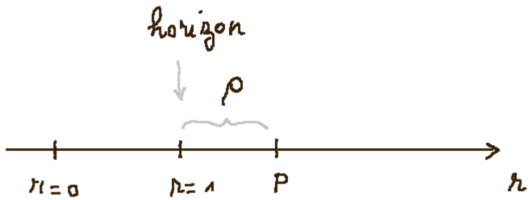

Figure 3 : $\rho =$ proper distance from the horizon.

If $(0,\ r,\ 0,\ 0)$ is the coordinate of P in the coordinate system $(t,\ r,\ \theta,\ \phi)$, notice that the proper distance between P and the horizon is not $r-1$. The quantity $r-1$ is a certain kind of distance but it is not the proper distance from the horizon.

How do we find the proper distance between P and the horizon ? Well, we know its differential when we move a distance dr along the r axis, figure 2 and equation (13). Therefore ρ, the actual proper distance from the horizon to P, is the integral

$$\rho = \int_{u=1}^{u=r} \sqrt{\frac{u}{u-1}}\, du \tag{14}$$

This is some function of r. The integral is a doable integral, but we don't need to calculate it. ρ is just some function of r, which we denote $\rho(r)$.

Moreover, the metric given by equation (11) can be expressed in terms of ρ instead of r, because r is some function of ρ. From equation (14), if we know r we know ρ, and if we know ρ we know r.

To make equations nice, let's also modify a bit the time. We define ω as follows

$$\omega = \frac{t}{2} \tag{15}$$

Now let's reexpress $d\tau^2$ of equation (11) with the variables ρ and ω instead of r and t. The second term $r/(r-1)\ dr^2$ is simply $d\rho^2$. We see this from equation (13), remembering that we decided to call the proper distance ρ.

Concerning the first term $(r-1)/r\ dt^2$, it can be expressed as a function of ρ times $d\omega^2$. For reasons that will become clear in a moment let's write this function $F(\rho)\rho^2$.

With this change of variables from $(t,\ r)$ to $(\omega,\ \rho)$, the metric giving $d\tau^2$ becomes

$$d\tau^2 = F(\rho)\rho^2 d\omega^2 - d\rho^2 - r(\rho)^2 d\Omega^2 \tag{16}$$

It is possible to calculate explicitely the last coefficient $r(\rho)^2$ but we don't need to. This form of the metric is sufficiently interesting as it is in equation (16), plus a little bit of knowledge about what $F(\rho)$ and $r(\rho)$ are like.

We already saw some of this in the last lesson, in the section entitled "Hyperbolic coordinates revisited" of chapter 6. But we are now going into more details. And we use now explicitely the proper distance which we denote ρ, whereas in chapter 6 we worked differently. We staid near the horizon, at $r \approx 1$, and worked directly with the quantity $r-1$, which we denoted ξ, see equations (26) to (28) of lesson 6.

Where is the horizon in terms of ρ? It is at $\rho = 0$, because ρ is the (proper) distance from the horizon.

In equation (16), we can say a number of things. First of all, when ρ becomes very big, far away, it is easy to show that

$$\lim_{\rho \to +\infty} F(\rho)\rho^2 = 4 \tag{17}$$

That just reflects the metric far from the black hole. That is one thing.

Next, let's look at what happens when ρ goes to zero. That is getting right up against the horizon. Again we easily establish that

$$\lim_{\rho \to 0} F(\rho) = 1 \tag{18}$$

The first term in the expression of $d\tau^2$, in equation (16), becomes $\rho^2 d\omega^2$. You may find that familiar. That was the reason, by the way, for defining the term function of ρ in front of $d\omega^2$ as $F(\rho)\rho^2$. Indeed, that way, close to the horizon $F(\rho)\rho^2$ is just ρ^2, or equivalently $F(\rho)$ is just equal to one.

Finally, let's look at the limit of $r(\rho)$ when ρ goes to zero. That we can work out. $\rho = 0$ means at the horizon. There $r = 1$, but we can do a little bit better. We can show that

$$\lim_{\rho \to 0} r(\rho) = 1 + \frac{\rho^2}{4} \tag{19}$$

This non standard way to write a limit means, if you prefer, that $r(\rho)/(1+\rho^2/4)$ tends to one.

Equations (17), (18) and (19) are basically everything that we need to know to study the metric.

Now the most important thing is to look at equation (16) when ρ is small. It is approximately

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2 - d\Omega^2 \tag{20}$$

That is exactly flat space in hyperbolic polar coordinates.

So let's review for a minute hyperbolic polar coordinates in the vicinity of a point H on the horizon, figure 4. There are two important quadrants in this picture, the right quadrant of events located farther than the horizon, and the upper quadrant of events located inside the horizon.

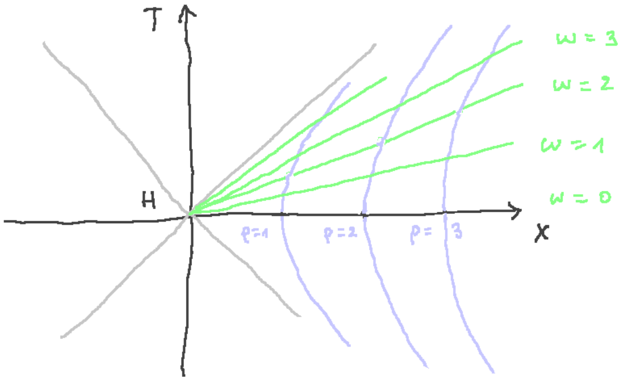

Figure 4 : Hyperbolic polar coordinates in the vicinity of the horizon.

Let's look first at the right quadrant. The first hyperbola corresponding to $\rho = 1$ is the trajectory in time of a particle fixed at one unit of ρ away from the horizon. The second hyperbola is the trajectory of a particle fixed at $\rho = 2$, etc.

That is just looking at the metric of equation (16) and noticing that, apart from the coefficient $F(\rho)$, near the horizon it just has the form of flat space in hyperbolic polar coordinates that we studied in chapter 4.

The lines of constant ω are the straight lines fanning out of the origin in figure 4, that is event H. We drew $\omega = 0$, that is the horizontal axis, $\omega = 1$, $\omega = 2$, $\omega = 3$, etc. Where is $\omega = +\infty$? It is right along the light-cone, which on the diagram is the line at 45°. Remember what is ω. It is just $t/2$. And we had defined t and r as follows

$$\begin{aligned} X &= r \cosh t \\ T &= r \sinh t \end{aligned} \tag{31}$$

So we have discovered some coordinates, $(\omega,\ \rho)$, in which time is like a hyperbolic angle. And time infinity is just the light cone at 45°.

What about the upper quadrant ? We already talked about it in the last lesson. It is the region where $r < 1$. When $r < 1$ the sign of the first two terms in equation (11), or equivalently in equation (16), interchange. In the upper quadrant ρ^2 is negative. We can see this from equation (19).

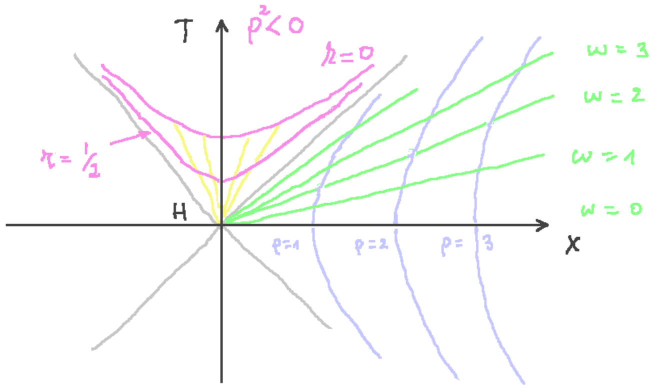

Figure 5 : Constant ρ's and constant ω's in the upper quadrant.

In the upper quadrant the curves of constant ρ, or equivalently constant r, are the hyperbolas shown in figure 5. We drew $r = 1/2$ and $r = 0$.

Nothing crazy physically is going on in the upper quadrant. Time has not become space, space has not become time. We have just introduced *coordinates* which have the funny property that when you go from the right quadrant to the upper quadrant, what was time-like becomes space-like and conversely. It is a complete coordinate artefact.

Yet there is something interesting about it. If we go far away from H, we have to remember that $F(\rho)$ is not just one. And $r(\rho)$ is not just one either. That means the metric along the ω direction, $F(\rho)\rho^2$, does differ from flat space. But the difference shows up in the way $F(\rho)$ varies as we move away from the black hole. Same comment on $r(\rho)$ in front of $d\Omega^2$.

The origin H in figure 5 is $\rho = 0$ but $r = 1$. What happens as we move up into the upper quadrant and r decreases? Eventually we hit the nasty point where $r = 0$. Then that is no longer just a coordinate glitch, we are at a real physical singularity.

If we calculated the curvature near the hyperboloid $r = 0$, it is not too hard to figure out what we would find. As r goes to zero, the sphere is getting smaller and smaller and the curvature is getting larger and larger. It becomes infinite on the hyperboloid $r = 0$. This is a true singularity. It is a place of infinite tidal forces, and it is basically a place where you wouldn't want to be.

The problem with the singularity is that once we are in the upper quadrant, once we are below the horizon of the black hole, we can't avoid it. It is not really a place. As we discussed in the last lesson, it is really, in a sense, a time. And, if we can avoid running into an obstacle in space – we just go around it –, we cannot avoid "running into the future".

If you are orbiting in a space station far away from the horizon of the black hole, as long as you don't do anything stupid, like jump off and allow yourself to get sucked in by gravity, you are safe in your capsule. Nothing bad will happen to you. But if you are foolish enough to think "I want to explore what is in there" and decide to go and see, as soon as you pass the horizon of the black hole, you are doomed. Not only there is no way you can get out, but there is even no way you can avoid flying into the singularity and being destroyed by tidal forces.

In order to get out you would have to exceed the speed of light. That corresponds to the 45° line in figure 5, or to the light cone of 45° lines if we add the other spatial dimensions. Once you are in the upper quadrant the best you can do to try to escape – if we assume that nothing can exceed the speed of light – is follow a 45° trajectory. And they all run into the hyperboloid $r = 0$ in a finite amount of your proper time.

Where is the point of no return ? It is actually the entire line at 45°. To understand why, let's ask ourselves : what is that light-cone ? In the right quadrant each hyperboloid is at a different value of ρ. And the entire hyperboloid corresponds to its value of ρ. In figure 5 we see the hyperboloids $\rho = 3$, $\rho = 2$, $\rho = 1$, etc. At the point H, $\rho = 0$. But in fact $\rho = 0$ all along the limit hyperboloid which is formed by the two lines at 45° and $-45°$.

There is something very different about this kind of geometry than ordinary geometry. Suppose we drew ordinary geometry, as in figure 6, and asked : where is the point $r = 0$?

Well we know that there is only one point where $r = 0$. It is the origin. Where is the point where $r = \epsilon$? It is not a point, it is a tiny circle around O. As ϵ goes to zero, the circle gets closer and closer to being just point O.

It is quite different in the geometry of space-time. Where is $\rho = \epsilon$? It is an entire hyperboloid close to the light-cone. As ρ gets smaller and smaller it tends to the light-cone, not just to the point H. And $\rho = 0$ is the horizon.

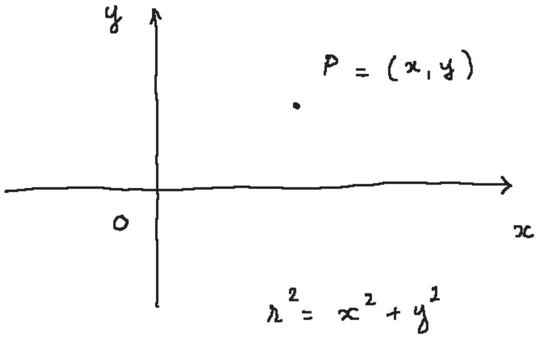

Figure 6 : Flat plane in ordinary Euclidean geometry.

So the horizon is not just the point H at the center. The horizon is all along the lines at 45° and −45°. Anybody who finds himself or herself beyond a point on these lines is inside the black hole, inside the horizon. And as we can see just looking at figure 5 such a person is doomed.

To summarize, the point of no return is indeed $r = 1$, or equivalently $\rho = 0$. But it is not only the point H. It is the whole line at 45°. The whole line in figure 5 is the horizon of the black hole.

Fundamental diagram

It is fundamental to understand the diagram of figure 5 very well in order to understand gravitational fields created by massive bodies, and eventually the theory general relativity. We reproduce it below.

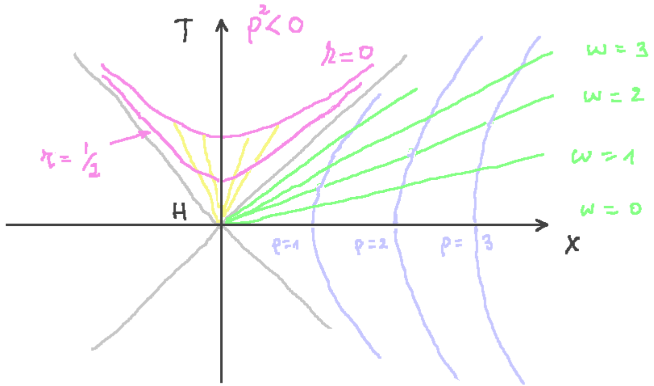

Figure 5b : Fundamental diagram to undertand the gravitational field created by a point mass in general relativity. The point mass is not at the point H, but at the curve $r = 0$. Point H and the two lines at 45° are the horizon of the point mass, called a collapsed star or a black hole.

Let's make miscellaneous comments on this diagram.

1. In the upper quadrant, the hyperbolas of constant r, or equivalently constant ρ, correspond to $\rho^2 < 0$. Therefore the ρ's are imaginary.

2. In the same upper quadrant, the straight lines coming out of H are still lines of constant t, or equivalently contant ω. But they are now space-like. And the hyperbola of a given r is time-like.

3. **What are the coordinates** $(X,\ T)$ **?** They are the coordinates of an observer who is near the horizon of the black hole, and is falling freely in the gravitational field of the black hole.

4. Someone falling freely through the horizon might indeed find it convenient to use the rectangular coordinates $(X,\ T)$ to chart what is going on around him or her. T is by definition the proper time of the observer. It is sometimes called the *coordinate time*. And X of course measures distances with the stick of the observer.

5. A particle which, unlike the observer, is at a fixed distance from the black hole, for instance at $\rho = 2$ from its horizon, follows in space-time a trajectory which, in the frame of reference of the observer, is the hyperbola in the right quadrant indexed with $\rho = 2$.

6. Time T and time t are different. Time T is the time of the observer. Time t is the time of any particle fixed relative to the black hole. Such a particle is living on one of the hyperbolas in figure 5b. More accurately, for each particle, its proper time is proportional to t, or equivalently proportional to ω which is $t/2$.

7. It is important to understand that, just like in Newtonian physics we need a reference frame to represent space and time – time being a neatly separate and universal coordinate –, in relativity we also need a reference frame to represent space-time. Since, due to massive bodies in the universe, there are no genuine uniform fields, no frame is clearly superior to any other. However the frame of reference of free-falling

observer is particularly convenient.

The coordinates $(\omega,\ \rho)$ have a name. They are called the Kruskal[3] coordinates.

History of black holes

When Schwarzschild wrote in late 1915 the Schwarzschild metric, he didn't know that the horizon was a horizon. As far as he, as well as Einstein and all the people who studied these things for a while, could tell, $r = 1$ was a nasty place where two coefficients in the metric given by equation (5) changed sign and also one of them – the one in front of dr^2 – went through infinity. They said "oh dear this coefficient is divergent at $r = 1$, something bad is happening there". And so the conclusion was that the horizon was some sort of singularity. In other words, the first people did not realize that the horizon was a smooth nonsingular ordinary place.

Sometime in the 1950's Finkelstein[4] realized that the horizon of a black hole was the point of no return, but that at the horizon itself nothing nasty happened to someone falling through. Only some finite proper time later would that person be anihilated at the singularity. He rediscovered coordinates that had already been written down by

3. Named after Martin Kruskal (1925 - 2006), American mathematician and physicist. Martin Kruskal had two brothers who also left a name in science : William, of the Kruskal-Wallis statistical test, and Joseph, of the Kruskal tree theorem.

4. David Finkelstein (1929 - 2016), American physicist.

Eddington[5] So they were called the Eddington-Finkelstein coordinates. They are not exactly the Kruskal coordinates, but are of the same type.

Martin Kruskal was not a relativity expert. He was a plasma physicist but very good at equations and very good at changing coordinates. He loved to change coordinates. Somebody showed him the Schwarzschild metric. He tried many changes of coordinates and eventually, in 1960, found these coordinates $(\omega,\ \rho)$ in which the metric has the nice form of equation (16), that is,

$$d\tau^2 = F(\rho)\rho^2 d\omega^2 - d\rho^2 - r(\rho)^2 d\Omega^2$$

And he proposed the now usual diagram that goes with them, figure 5b.

Before being called a black hole, this kind of object was called a collapsed star or massive collapsed star. The term black hole was coined by John Wheeler[6].

John Wheeler was a very nice and very sweet man. And he was a good friend of mine too. He was politically very conservative, contrary to me. His political conservativeness had to do with one thing and only one thing : he was concerned about the Soviet Union having any nuclear weapons. So he was very anti Soviet Union and especially anti Soviet expansionism. We would argue about it, not so much me, but some of my friends. He was however a very thoughtful, gentle person.

5. Arthur Eddington (1882 - 1944), English astronomer, who headed the famous expedition that, on the occasion of the solar eclipse of May, 29 1919, first confirmed Einstein theory of general relativity.

6. John Wheeler (1911 - 2008), American theoretical physicist.

He did not have a socially conservative side. His political bend did not extend to social issues. In fact I remember once we were sitting in a café in Valparaiso, Chile, John – he was like 85 years old at the time – my wife and I. While sitting he started to look very agitated. I said : what's the matter John, do you feel well ? He said : no, I want to get up and take a walk. I said : where are you walking ? Do you want me to take a walk with you ? He said : no, no, I'm gonna take a walk by myself. I asked : what are you going to do John ? He answered : I want to check out the bikinis. So he was not a social conservative.

In his first paper on the Schwarzschild metric he coined the term black hole. That caused a stir. The Physical Review did not want to publish it. It was before my time as an active scientist actually, or just about, but I knew there had been a problem. I didn't know what it was. I learned it was not just the Physical Review being its usual conservative self. No, they were being hyper conservative : they thought that the term black hole was obscene. So they refused to publish the paper at first. And he fought and fought and fought with them, and eventually won. Then just to get back at them he entitled his next paper "black holes have no hair".

Incidentally, what does it means for black holes not to have any hair ? It means that if you take a non-rotating black hole, gravity is so strong that it will always pull it together into a sphere. Even if it starts very asymmetric, like two rocks coming together, after a very small amount of time the horizon will pull itself into a sphere, and become indistinguishable from a perfect sphere. And John called that characteristic of not having any visible structure nor any visible defects on it, the property of not having hair. So the statement that the black holes have no hair is just the statement that gravity is so strong that it will pull the horizon of any black hole into a perfect sphere. If it is rotating then the sphere can get deformed. It can get into an oblate spheroid. But the nature of the oblate spheroid depends only on the angular momentum. So black holes have no hair.

Now, let's talk about things or people falling into a black hole.

Falling into a black hole

Back at the diagram in figures 5 or 5b, you might ask what do the left and bottom quadrants represent. We are going to see that this other half of the diagram, the half below the $-45°$ line, has no real significance. We will come to it. But for the moment we are mainly interested in the upper right hand part. The exterior of the black hole is the right quadrant, and the interior of the black hole is the upper quadrant.

Let's redraw the picture, figure 6. It is Alice's turn to fall in. For simplicity of notations, we use coordinates $(t,\ r)$, which are equivalent to $(\omega,\ \rho)$, the close correspondence being given by equations (14) and (15). Bob stands somewhere outside the black hole at a fixed position. Therefore his trajectory is a hyperbola of constant r as his proper time, proportional to t, ticks.

The horizon is $t = +\infty$. It is very strange but this place, that we thought of as being a place, also has the character of being a time. If you prefer, it is where $\omega = +\infty$, since ω is half of t.

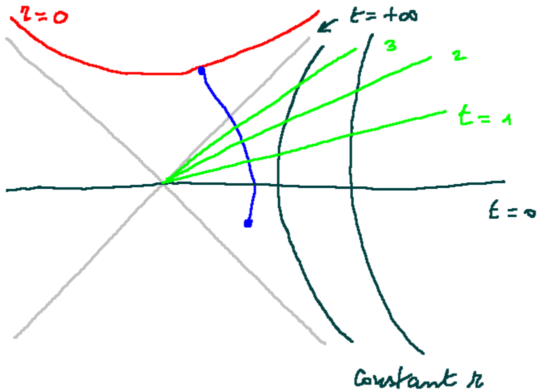

Figure 8 : Alice falling into the black hole.

Looking at the diagram, we can see that while Alice is falling in, she doesn't pass the horizon until t equals infinity. It is in that sense that an in-falling object never passes the horizon. It is a statement *in Bob's time*. And indeed Bob cannot see an object nor a person fall through the horizon.

But of course Alice does pass the horizon. We can see her in the diagram sailing into the singularity. Nothing crazy happens to her as she passes the horizon. It is just the strange set of coordinates that says that $t = +\infty$ when Alice crosses the horizon of the black hole.

So you might think : well, it doesn't really mean anything to anybody the fact that Alice never gets in. But on the other hand Bob is out there, staying at a fixed position, that is, on a hyperbola of constant r. And Bob is watching Alice. Let's examine more precisely what he sees of Alice, what does that mean to see Alice. How does he see Alice ?

To "see Alice" at time t means for Bob to receive a light-ray emitted by Alice sometime earlier [7], and arriving at Bob at time t. In other words, Bob "looks into the past". For instance, at time t_1, when Bob is at point P_B, he sees Alice at point P_A. At time t_2, when Bob is at point Q_B, he sees Alice at point Q_A, figure 9. Remember that light travels at 45° angle.

Figure 9 makes it clear that Bob will never see Alice cross the horizon, that is, the line $t = +\infty$. As long as Bob stays outside the black hole and looks back, what he will see is Alice getting closer and closer to the horizon, but never passing it.

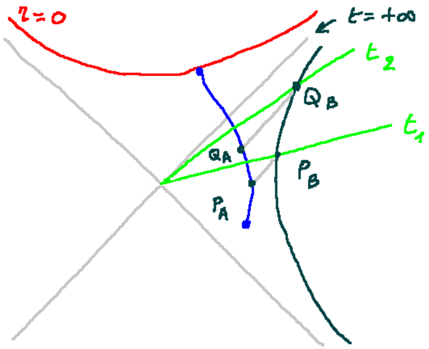

Figure 9 : Bob sees Alice.

7. The lines of simultaneous time for Bob are the straight lines fanning out from the origin of the diagram.

So in fact, you may say that it is not just a coordinate artefact that Alice "never crosses the horizon". It is physically what Bob (doesn't) see. As far as Bob is concerned Alice never passes the horizon.

If Alice in her own frame of reference passes the horizon, she cannot send a signal to Bob warning him "Hey Bob, I just passed the horizon". Why ? Because such a signal would travel on a 45° line in figure 9 and never reach Bob.

Everything that Bob can observe and measure, his entire physical observations, his entire universe, all involve Alice outside the horizon. As far as he knows, her heart slows down as she approaches the horizon. It somehow stops beating, in the sense that each beat takes longer and longer. So for him, she all but dies at the horizon – in an infinite amount of time. He can have no idea about what happens to her past the horizon, because for him it is beyond the end of times. And she is indeed doomed then...

One important point ought to be mentioned : we are considering a very big black hole. Indeed, we assume that the tidal forces at the horizon are negligible. The apparent squashing of Alice at the horizon, viewed by Bob, has nothing to do with tidal forces. It is only a variety of Lorentz-FitzGerald contraction already studied in the last lesson. No tidal force related deformation happens to Alice at the horizon.

For a collapsed star with the mass of the Sun, the Schwarzschild radius is approximately 3 kilometers, and the tidal forces at the horizon would already be huge. The Sun when it collapses will make a small black hole, a black hole of the nasty type as we saw. On the other hand at the center of our galaxy, there is a humongous black hole, which is rather mild even close to its horizon – but of course you would be ill-advised to go in and explore.

When Bob watches Alice, he sees her slow down. Even her heartbeats slow down – rather intrepid for somehow about to fall into a black hole! Does this mean that Alice sees something special happening to Bob when she watches him?

Let's study what Alice sees when she looks at Bob. Again all the answers are given by the diagrams. See now figure 10.

As Alice sails along her trajectory she sees light-rays emitted by Bob sometimes in her past with no problem. When she is at R_A, outside the horizon, she sees Bob at R_B. When she crosses the horizon at S_A, she sees Bob at S_B. Until she hits the singularity, where she will be anihilated, she continues for a while to see Bob without difficulty.

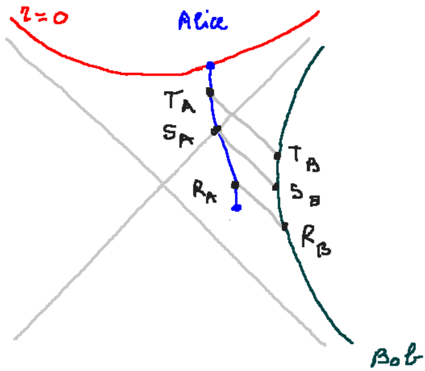

Figure 10 : Alice sees Bob.

Moreover there is nothing indicating that she would see Bob speed up or slow down.

Alice can still see Bob, when Bob can no longer see her. And Alice doesn't see anything particular happening to Bob. To her, everything is normal as usual. Of course Alice cannot see Bob past a point a little after T_B. But that is only because, by then, the tidal forces will have torn her apart and terminated her.

In conclusion, there is a total asymmetry between Bob and Alice.

Exercise 1 : Use the diagram in figure 5 to describe what a third person, say Charlie, following Alice sometime later would see of Alice and of Bob at different times.

In exercise 1, so long as Charlie or Alice is still outside the black hole, Bob can receive a message from him or her. And, for Bob, the "last" messages will be infinitely late in time. So he does not see either of them ever cross the horizon.

With the diagram, you can also analyse how Alice and Charlie are able communicate with each other.

In the case that Alice and Charlie are both in free fall, their trajectories will be parallel and look almost like straight lines.

However they fall, you can think of Bob being eventually unable to see them as a consequence of the fact that he has to accelerate [8] to stay outside the black hole.

In all cases, what one of the protagonists can see of the others is simply and strictly related to the pair of points, in their respective trajectories, that 45° light-rays can link.

In short, the diagram says it all.

Miscellaneous notes

Here are miscellaneous notes from a question and answer session with students :

- There are some movies that purport to show what is happening when someone falls into a black hole. Andrew Hamilton has made some simulations of what it looks and feels like to fall into a black hole [9].

- We said that nothing special happens at the horizon but of course the light that is coming at you is coming in very peculiar and funny ways.

- The movies are not very illuminating. You won't get much out of them but it might be fun to watch one.

8. He has to accelerate toward the exterior *viewed in their frame or frames of reference*. Remember that the equivalence principle says that being at rest and feeling gravitation in frame 1 is equivalent, viewed from a frame 2 with no gravitation, that is, in free fall in the uniform gravitational field of frame 1, to being accelerated upward.

9. See for instance on YouTube "Journey into a realistic black hole", by Andrew Hamilton.

They really are disorienting. If you are in an auditorium and one of them is projected on a big screen in front of you, they can make you nauseous and seasick.

- You might wonder what happens at the origin in the diagram of figure 5, that is at the point H on the horizon. Answer : nothing much because in fact for a real black hole, that is formed for example by stellar collapse or something like that, the center part of the diagram isn't even on the figure. Only a portion of the diagram in the upper and right parts is relevant.

- Figure 5 is that of an idealized black hole. For a real black hole which forms by collapse only a portion of the diagram, which does not include H, means anything.

- As a black hole gulps a mass coming from the outside it own mass gets larger. Therefore the Schwarzschild radius gets larger too.

- The pictures we drew are appropriate only when the objects or persons falling in are much lighter than the black hole. So the black hole doesn't react strongly to them.

- If a big mass, of a size not negligible compared to that of the black hole, falls into it, when the mass approaches the black hole, the horizon of the black hole will bulge to merge with the coming mass, see figure 11.

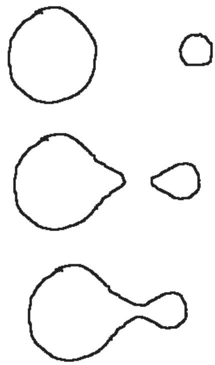

Figure 11 : Black hole (left) gulping another sizeable mass (right).

- Then because black holes have no hair, it will quickly pull itself together back into a sphere. But the sphere will be slightly larger.

- The process in figure 11 can be calculated with Einstein field equations, which we will study in chapter 9.

- How can we see the black hole at the center of the Milky Way ? Answer : we don't "see" the black hole itself. We identify phenomena that are due to the presence of the black hole.

- The light that we see from a black hole is not coming from the horizon. It is coming from all sorts of stuff, hot stuff, that is circulating around the horizon and that is heated up by energy and collisions. Remember for example that at a certain distance around the black hole we can have photons of light moving around in circles. All sorts of very complicated collisions can take place around the black hole, which sends out particles and light some of which ends up in our eyes or measuring apparatus.

- In figure 5, we were talking about the idealized situation of a black hole in empty space. But in reality there are all kinds of stuff around it. There is an accretion disk. There are plenty of things falling in. And what you see is not coming precisely from the horizon. It is coming from all this activity at some distance from the horizon.

- The biggest black hole that is known has a mass something like 10 to the 9th solar masses.

- We haven't talked about what happens inside the black hole when a mass is absorbed. Next chapter, we are going to talk exactly about how a black hole is formed, how the horizon forms, and if you throw in more material how the horizon responses. We will examine a simple example, and see that things are different from expected. They are surprising and yet logical.

- Does anything change if Bob is orbiting the black hole ? No, not much. It becomes a more complicated problem to be quantitative about exactly what he sees. But there is no fundamental change.

- A black hole is the analog in relativity of a point mass in Newtonian physics. All the mass is at the center. There is no such thing as getting deeper and deeper into the mass of the black hole after we pass the horizon, like what happens when we enter below the surface of the Earth and the gravity caused by the outer layers for us gets out of the equations.

- The vast majority of things or collections of things in the universe have some angular momentum. As a consequence the vast majority of black holes rotate fast. Why ? Because in the process of collapsing and forming a black hole, even if at first the material does not show much rotation, it has some angular momentum. Then, as its dimension shrinks, like an ice skater pulling his or her arms along the body, it will spin faster and faster.

- Stars form when materials agglomerate and start to radiate. The interior of the star burns hydrogen or helium or whatever. When the star eventually runs out of fuel, the forces, that is, the radiation pressure that is preventing it from collapsing disappears and the star begins to contract.

- Depending on how heavy it is, it might contract into a white dwarf, that is a more less ordinary thing, pretty dense but still made out of nuclei of atoms and so forth. If it is heavier it might collapse into a neutron star. That is a very compact object, but it can still support itself because the material is strong enough to support itself against gravity. If it is still heavier then it may contract past its own Schwarzschild radius. Once it falls past its own Schwarzschild radius, it is doomed : it becomes a black hole.

- The Sun, when it eventually "turns off", will not form a black hole. If nothing else happens to it, like merging with something else, it will form a white dwarf.

- In theory black holes can "evaporate" and disappear. But the process is very long. For instance a black hole with the mass of Mount Everest would take longer than the age of the universe to evaporate.

- Aside from gravity collapse that we just described, another process of formation of black hole is via violent collisions. Velocity can replace gravity and slam stuff together. If there is enough violence a black hole can form – even though there was not enough material for gravity collapse.

- The smallest imaginable black hole is a black hole of Planck mass [10]. By slamming particles hard enough you could theoretically make small black holes.

- Similar collisions happen naturally, with higher energies than in man-made accelerators, when cosmic rays hit the Earth upper atmosphere. Therefore possible artificial black holes, created for instance at the LHC in Geneva, would be no more dangerous.

The study in more detail of the formation of black holes is the subject of next chapter.

10. $m_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G}} \approx 1.22\text{x}10^{19} GeV/c^2 \approx 2.18\text{x}10^{-8} kg$