Lesson 8 : Formation of a black hole

Notes from Prof. Susskind video lectures publicly available on YouTube

Introduction

In the last lesson, we explained the Kruskal coordinates – also called Kruskal-Szekeres [1] coordinates – for a black hole. We shall go over them again, for practice. Then we shall use a different type of diagram to represent the geometry of space-time near a black hole, and see what other phenomena it reveals. Finally we shall examine the formation of real black holes.

Kruskal–Szekeres coordinates

Remember the fundamental diagram :

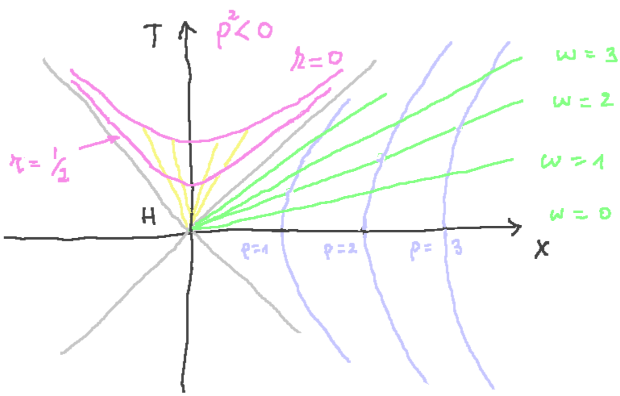

Figure 1 : Kruskal diagram near the horizon of a black hole.

1. George Szekeres (1911 - 2005), Hungarian–Australian mathematician.

In the last lesson we saw that the geometry of a Schwarzschild black hole near the horizon looks like flat space [2]. We build near the horizon hyperbolic polar coordinates which illustrated that points at fixed distances outside the horizon behave as if they were fixed in a uniformly accelerated coordinate frame.

We remember that somebody standing in the right quadrant is outside the horizon of the black hole. Somebody in the upper quadrant is inside the horizon of the black hole.

The 45° lines represent the motion of light-rays either going to the right or going to the left. We saw that not only point H but the entire lines at 45 ° form the horizon.

We are going to see that the left quadrant and the bottom quadrant don't have any physical significance for a real black hole. It is only the right quadrant and the upper quandrant which have real significance.

Outside the black hole, somebody at rest, that is, at a fixed distance r, or equivalently at a fixed distance ρ from the horizon [3] – with a trajectory on a hyperbola –, is of course experiencing acceleration.

This is the same kind of acceleration which you and me experience even when we are standing still. Our weight is due the gravitational field of the Earth, and is equivalent to being constantly accelerated.

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2. In fact, it looks locally like flat space anywhere in space-time except at the singularity, just like any smooth surface in Euclidean space looks locally like a plane.

3. ρ is the proper distance. When $r > 1$, ρ is related to r by a simple increasing function which we studied in chapter 7.

On the surface of the Earth, we represent the geometry of space and time with a Euclidean space plus a universal time coordinate. *We reason on only one dimensional spatial axis*, which is radial from the center of the Earth. A uniformly accelerated frame is simply that of an observer inside an accelerated elevator. Various points fixed above each other in the elevator, viewed in the reference frame of the stationary observer resting on Earth, have trajectories which are parallel lines when time varies. Lines of fixed time don't converge to a center point but are parallel too, orthogonal to the trajectories. And the time coordinate is the same in any reference frame.

In relativity the geometry of space-time is different. Space and time are intimately related. Simultaneity, for instance, is frame dependent. However, a uniformly accelerated reference frame, far away from the origin of the Minkowski diagram, is very much like an ordinary accelerated reference frame near the surface of the Earth : trajectories are almost parallel vertical lines (if the time axis is vertical), and time and proper time are almost the same.

We are so accustomed to the acceleration which we permanently feel at the surface of the Earth that we tend to forget it. But somebody who has been in free fall in outer space for too long, lost the sense of what a gravitational field is like. If you place him or her on the surface of the Earth that person will be very much aware of a new feeling. The person will be experiencing a sense of upward acceleration.

Spread on a radial axis jutting out of the horizon of the black hole (angles θ and ϕ play no role, they are constant), we then have a collection of people, at proper distances respectively $\rho = 1$, $\rho = 2$, $\rho = 3$, etc. The closer to the horizon a person is, the stronger the acceleration he or she feels.

In figure 1, the coordinates T and X are those of a free falling observer, who is anywhere in space-time except at the singulariy $r = 0$, and happens to look, in his or her coordinate frame, what is going on near the horizon of the black hole.

The observer sees the collection of persons at the various ρ's as being at fixed positions in a uniformly accelerated reference frame. In chapter 4, we defined what is a uniformly accelerated reference frame in relativity.

Pay attention to the fact that in such a uniformly accelerated reference frame, the actual acceleration – the physical push felt by each person – is not the same at different ρ's. The smaller ρ is, the stronger is the acceleration. This is shown by the growing acuteness of the hyperbolas near H. On the other hand, the acceleration stays the same along a hyperbola of constant ρ.

In the right quadrant, the straight lines fanning out of H are the lines of constant proper time t or ω for the people in the accelerated frame [4]. As ω tends to $+\infty$, the lines become closer and closer to the 45° line of a light-ray emitted from H.

In the above eight paragraphs we described the outside of the black hole.

4. Remember that ω is simply $t/2$.

The inside of the black hole is represented in the upper quadrant.

In this upper quadrant, the is a point where the geometry of space-time is no longer equivalent to locally flat. There is something nasty happening. It is at the singularity [5]. Strangely enough, the singularity is represented by a whole curve in figure 1 [6]. It is the hyperbola marked with $r = 0$. The tidal forces are very strong near the singularity.

The problem for an adventurous person exploring space-time is that once you find yourself inside the horizon of a black hole, that is, in the upper quadrant of figure 1, there is no way out – unless you could exceed the speed of light. And you will inescapably hit the singularity in a finite amount of your own time. So you are doomed.

In summary, outside black holes you or objects or light-rays can get into the black hole, or they can stay out. But inside the black hole nothing can get out, and everything ends up at the singularity in a finite amount of time.

That is all there is to know about the geometry of black holes. It is not just equivalent to plain flat space. The geometry is warped because of the singularity.

5. Notice that we would have the same kind of nasty singularity for the Earth, even in Newtonian physics, if we considered that all its mass was concentrated at its center.

6. There is no longer any analog in Newtonian physics. This is due to the metric of space-time having three positive signs and one negative sign. Furthermore the Schwarzschild metric is such that the radial space and the time coefficients exchange signs at the horizon. Therefore the trajectories for $r < 1$ are space-like and no longer time-like as usual.

Penrose diagrams

It is very convenient to redraw the diagram in figure 1 differently. We are going to end up with Penrose coordinates and a Penrose diagram. But these are built in several steps. So let's take the steps one by one.

In figure 1, space is infinite. Time may come to an end at the singularity, but the light-like directions off at 45° are also infinite. Therefore we cannot draw the entire space-time in the limited dimensions of the page.

For many purposes it is useful, however, to be able to draw the entire space-time on a finite sketch. It provides some visual tools and helps intuition.

Let's try doing that but let's start with good old flat space-time. We will do a coordinate transformation which pulls the whole thing into some finite region of the page.

This is useful, incidentally, for geometries in space-time which display rotational symmetry. Rotational symmetry, or invariance, means that what happens at any event in space-time doesn't depend on the direction of the event from some center.

Let's focus for a moment on ordinary 3D Euclidean space. To describe a system which displays rotational symmetry, it is usually useful to use ordinary polar coordinates. And when we have rotational symmetry, we often don't need to worry too much about the angular direction. Aside from time, the other coordinate that matters is the radial distance.

Therefore, to start with – before shrinking the whole space-time on the page –, we think of ordinary flat space-time as having a time axis T and a space axis R, which is a radial direction, see figure 2. The time goes from $-\infty$ to $+\infty$. And the spatial coordinate R goes from 0 to $+\infty$.

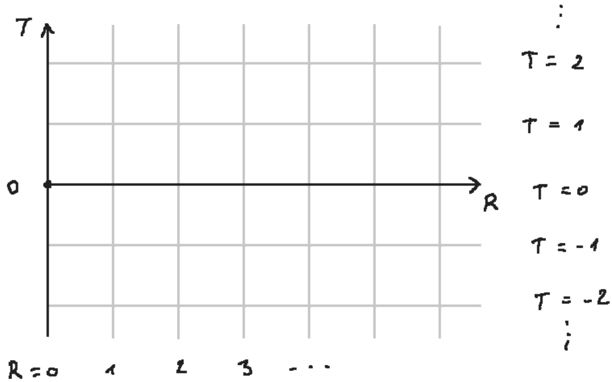

Figure 2 : Flat space-time.

We put some markers : $T = 0$, $T = 1$, $T = 2$, $T = -1$, $T = -2$, etc. And $R = 0$, $R = 1$, $R = 2$, $R = 3$, etc. The entire space-time is not yet shrunk on the page. So far T can go up to heaven, and down to hell. Similarly, the radial axis R doesn't have any limit on the right.

In all cases, we will imagine using units so that the speed of light is $c = 1$. For instance, if we use seconds for time, then the space units will be light-seconds.

Then some light-rays follow 45° lines. But not all of them. Light-rays coming from the past *aimed at the spatial origin* and then going on into the future, form a pair 45° half lines as shown in figure 3.

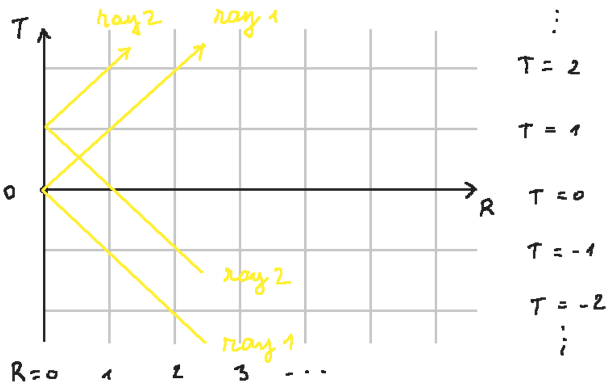

Figure 3 : Two light-rays aimed at the spatial origin.

Another picture will help us understand why some light-rays follow paths like in figure 3, and some don't. Figure 4 shows space in polar coordinates centered on the same point as above. The time axis is not represented.

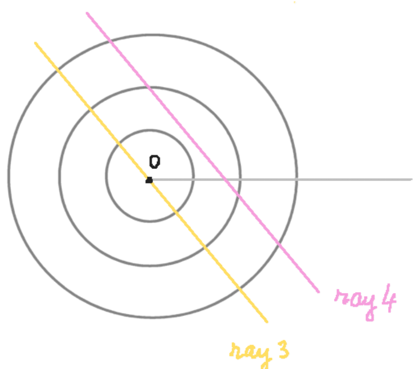

Figure 4 : Two light-rays, one aimed at the center, and not the other.

In figure 4, ray 3 is aimed at the center. That ray, represented in figure 3, would appear to bounce against the vertical line $R = 0$, like ray 1 and ray 2.

But ray 4, if we represented it in figure 3, would not bounce against $R = 0$. It would be coming from far away in the past, from what we call light-like infinity. As long as it is far away, unless we have very good instruments, we could not say whether it would hit the center. But near the vertical line $R = 0$ it would swerve before hitting it, see figure 5.

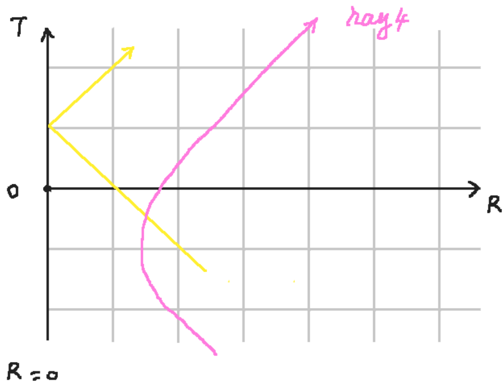

Figure 5 : Light-ray 4 passing by the origin, but not hitting it.

Light-ray 4 doesn't hit the origin because it was not aimed at the origin. Far away from the small radiuses it follows almost 45° lines in figure 5. In fact the trajectory of ray 4 is a hyperbola (but that has nothing to do with the hyperbolas of the Kruskal diagram).

In figure 5, light-ray 4 looks like it is repelled from the origin. Of course it is not repelled from anything, it just passes by near the origin before flying off again, all in a straight line as shown in figure 4. But the phenomenon appearing in figure 5 really is what we call centrifugal force. It is the centrifugal force that keeps the light-ray from hitting the origin.

We went over these things in figures 2 to 5 to get some landmarks and prepare for the next step.

The next step is to squeeze the diagram in figures 2, 3 and 5, to fit the entire space-time on the page. In particular what we dubbed the *light-like infinity* will now appear as a point somewhere on the graph.

Of course the diagram will be deformed. It is not going to look the same. But we are going to keep one feature fixed : light-rays will keep following 45° lines or asymptotes. That is a useful thing to do because then we can see how light rays move, and we can see what is going slower or faster than the speed of light.

The shrinking of the whole space-time into a limited diagram is done mathematically in two steps as follows.

We introduce a first set of new coordinates.

$$\begin{aligned} T^+ &= T + R \\ T^- &= T - R \end{aligned} \tag{1}$$

In this set of coordinates, lines of constant T^+, for instance, are diagonal lines at $-45°$ angle, see figure 6.

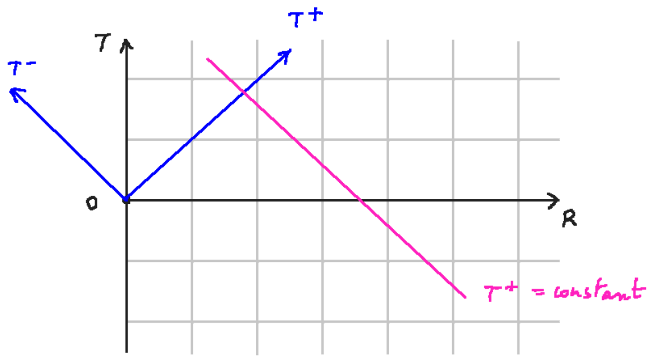

Figure 6 : Coordinates $T^+ = T + R$ and $T^- = T - R$.

Similarly, lines of constant T^- are diagonal lines at $+45°$.

So now we have to sets of coordinates to describe our flat space-time. We have $(T,\ R)$ and we have $(T^+,\ T^-)$.

Let's look at the vertical line going through O. It is $R = 0$ in the old coordinates $(T,\ R)$. And it is easy to see that in the new coordinates$(T^+,\ T^-)$, it has the equation

$$T^+ = T^- \tag{2}$$

One way to see it is to notice that it is the usual equation for the line splitting the angle formed by the two axes T^+ and T^- in half. Or we can write $T^+ = T^-$ as $T+R = T-R$, which becomes $2R = 0$, or equivalently $R = 0$.

For obvious reasons, the coordinates $(T^+,\ T^-)$ are called light-like coordinates.

Now we introduce a second set of new coordinates, so as to shrink the whole plane $(T^+,\ T^-)$ into a bounded area of the page. The second set is U^+ and U^-.

U^+ will be an increasing function of T^+ such that when T^+ goes from $-\infty$ to $+\infty$, the coordinate U^+ goes from -1 to $+1$, see figure 7.

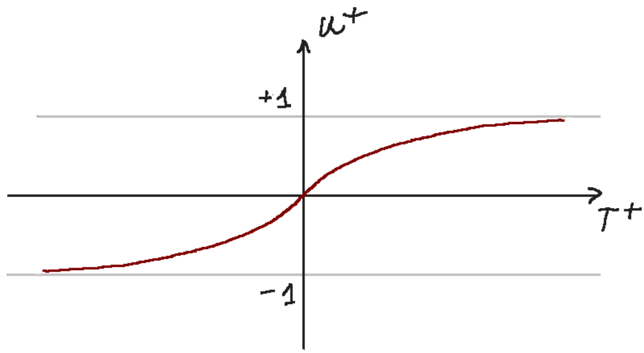

Figure 7 : U^+ as a function of T^+.

There are plenty of such functions. The one that is usually used is hyperbolic tangent. And we will apply the same transformation to T^- as well. So we define

$$\begin{aligned} U^+ &= \tanh T^+ \\ U^- &= \tanh T^- \end{aligned} \tag{3}$$

You don't need to know much about hyperbolic tangent, except that is has a graph as in figure 7.

Now we represent the flat space-time in the coordinates $(U^+,\ U^-)$. When T^+ goes to infinity, U^+ never gets bigger than one. Same for T^- and U^-. Figure 6 becomes the following figure 8.

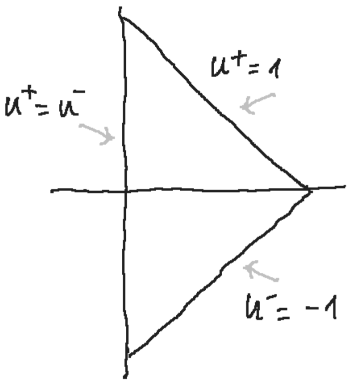

Figure 8 : Space-time in coordinates $(U^+,\ U^-)$.

All we've done is squish the geometry of figure 6, vertically and horizontally, onto a finite triangle. This will become clearer when we look at light-rays.

Let's look at an in-going light-ray aimed at the origin. It corresponds to

$$T^+ = constant$$

After hitting the origin, it becomes an outgoing light-ray from the origin. It then corresponds to

$$T^- = constant$$

The coordinate transformation has the property that such light-rays are still straight lines as shown in figure 9.

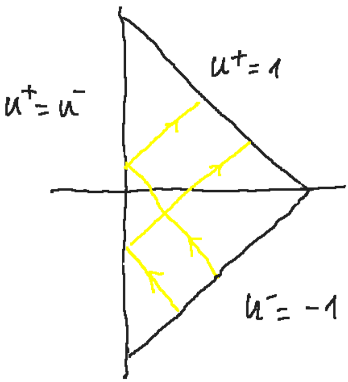

Figure 9 : Light-rays going through the origin,
in $(U^+,\ U^-)$ coordinates.

Another enlightening graph is that of constant times, see figure 10.

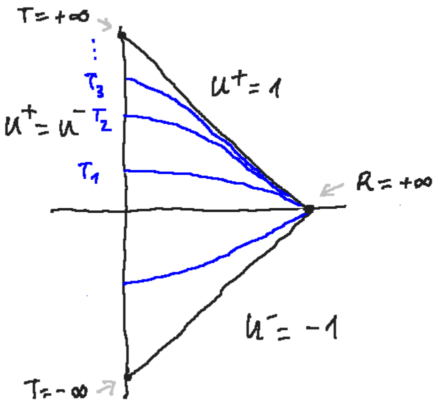

Figure 10 : Lines of constant times.

We have squished "spatial infinity" into the point $R_{+\infty}$. It is called *space-like infinity*. And as we look at bigger and bigger fixed times, the corresponding curves will be closer and closer to the line $U^+ = 1$.

Now let's look at fixed spatial positions. They will all go to $T_{+\infty}$, coming from $T_{-\infty}$. They are shown in figure 11.

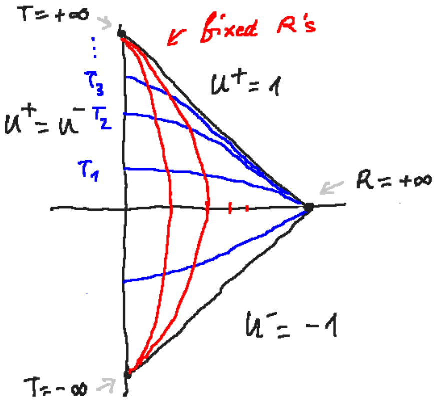

Figure 11 : Fixed position R,
in $(U^+,\ U^-)$ coordinates.

The diagram of figure 11 is "the same" as that of figure 5. But we have operated a change of coordinates so that every point which had some coordinates $(T,\ R)$ gets mapped somewhere onto a finite triangle.

The process of bringing everything from figure 5 into a finite diagram (fig 11), without changing the way radial light-rays trajectories appear (fig 9), is called *compactification.* And figure 11 is called a *Penrose diagram* or a *Carter*[7]*-Penrose*[8]

7. Brandon Carter (born 1942), Australian theoretical physicist.
8. Roger Penrose (born 1931), English mathematician and physicist.

diagram because actually the Australian physicist Brandon Carter invented it first.

That closes our section on the Penrose diagram of a flat space-time. Now we turn to black hole geometry. And this will lead us to a discussion of wormholes.

Wormholes

We succeeded in representing the entire flat space-time on the page. Now the question is : can we apply the technique to black hole geometry ?

Some terminology to start with : the right tip of the triangle in figure 11, where $R = +\infty$, is called spatial infinity, or space-like infinity. The tip of the triangle on top, along the right side, where $T = +\infty$, is called future time-like infinity. And the tip at the bottom is called past time-like infinity.

The 45° line between $T_{-\infty}$ and $R_{+\infty}$ is where all light-rays come from (fig 9). And the line between $T_{+\infty}$ and $R_{+\infty}$ is where they all go, after hitting the origin of swerving past the origin. The standard notation for the segment $T_{-\infty}$ to $R_{+\infty}$ is $\mathcal{I}^-$, read "script i minus", or sometimes "scry minus". And the segment $T_{+\infty}$ to $R_{+\infty}$ is $\mathcal{I}^+$, read "script i plus", or "scry plus". They are called respectively past light-like infinity and future light-like infinity. They are the places where light-rays begin and where light-rays end. This is all summarized in figure 12.

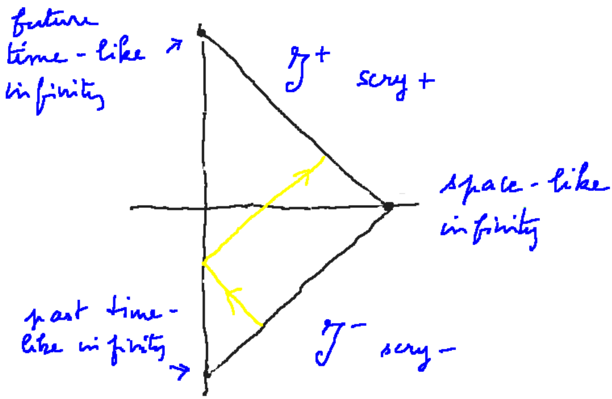

Figure 12 : Flat space-time compactified.

Figure 12 is what flat space-time looks like when compactified. Now what about a black hole. What does that look like ?

We can do exactly the same thing. We can take the entire diagram in figure 1 (where we can rename the X-axis the R-axis), that is, the original Kruskal coordinates and diagram. We can again introduce light-like coordinates, $T^+ = T + R$ and $T^- = T - R$, shrink them into functions varying between -1 and $+1$. We do exactly the same operations. What will we get ?

What it will look like is the diagram in figure 13. Again it shrinks the original four quadrants into four squares, but the upper square is halved by the singularity. And we disregard momentarily the lower quadrant.

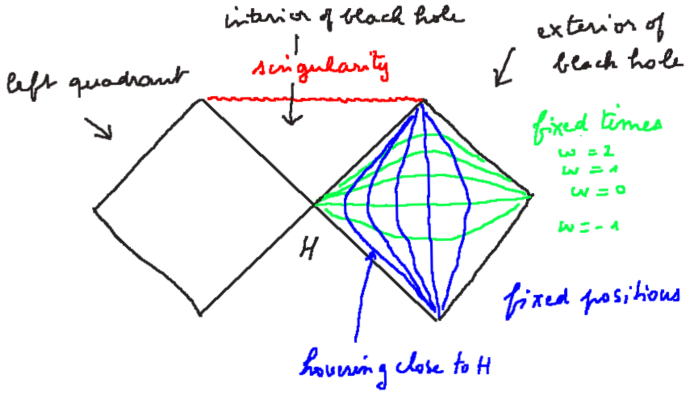

Figure 13 : Black hole space-time compactified.

The exterior of the black hole is the right square. The hyperbolas of fixed positions hovering above the horizon H become the blue curves. The original straight lines of constant time ω become the green curves.

The interior of the black hole is the upper part. It is the place where we are doomed. It is the half square bounded by the singularity. The singularity is not off at infinity. Remember that once we are inside the horizon, we cannot stay at a fixed position on the graph because we cannot avoid the future, and it takes a finite amount of time to travel to the singularity and be anihilated by infinite tidal forces.

Figure 13 shows the entire space-time of a black hole in a Penrose diagram. If we are in the right part we can stay where we are, but that takes acceleration in the sense that we have to oppose gravitation pulling us in. We can even escape to farther away – but, remember, all outer space is now represented in the right square. We can also fall beyond the horizon. Once we are inside the horizon, we cannot go back to the right part. We are doomed.

The picture in figure 13 asks an obvious question : what could possibly correspond to the left square ? Same for the lower part. These are the compactified representations of the left quadrant and the lower quadrant of figure 1.

The left quadrant and the lower quadrant have no real meaning for a real physical black hole. We will see that when we work out a real physical black hole, how it forms, etc.

Nevertheless, we can ask ourselves : what kind of geometry is described by the full extended Kruskal-Penrose diagram of figure 13. It seems to have two exterior regions – the right and left squares – connected together at the horizon H. Remember that, at the horizon, $R = 2MG$. This value of R is also denoted R_S, and called the Schwarzschild R. It is the place where $(1 - 2MG/R)$ changes sign. We arrive from above, that is from the exterior region, the coefficient changes sign, and we continue in the interior region, represented in the upper quadrant or upper half square.

Now suppose, at a fixed time, say $T = 0$, we look at a slice of space, or at the entire space. What does it look like ? In figure 13, it is just the complete horizontal segment from the right to the left of the figure. But in all our diagrams so far, for the sake of drawing, space was only one-dimensional. Think of space as three dimensional. Then for each R it is the surface of a sphere – also called a 2-sphere – of radius R.

When we start far away, the celestial sphere is very big. Then, as we approach H, the 2-sphere shrinks to $R = 2MG$.

If, for drawing purposes, we think of space as only two-dimensional, for each R the 2-sphere becomes a "1-disk", that is, the boundary of a disk, in other words a circle. Flipping the diagram from horizontal to vertical with H down below, we can represent these circles in figure 14.

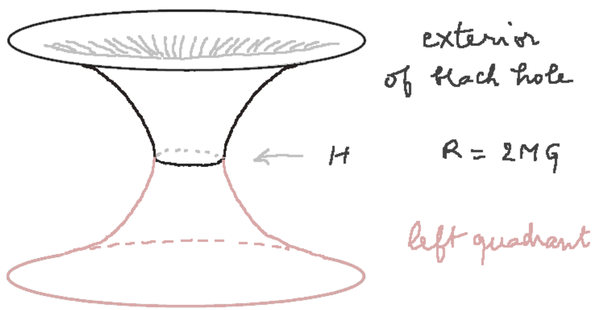

Figure 14 : Wormhole.

On the diagram, at the horizon, instead of turning into the interior region (upper quadrant or upper half square), we just "kept going". That is the lower part of figure 14. It represents the left quadrant, or left square, of the previous figures.

Once we have passed the horizon in this way, the space of each R starts expanding again. It looks like we can pass through from one side to the other by going through what people call a *wormhole*. It is also called an *Einstein-Rosen bridge*.

It connects two seemingly external regions of the black hole, which get bigger and bigger as we move away from the bottleneck. In other words, it looks like the black hole is connecting two universes, or two asymptotic regions.

You might think : well we could pass through the bottleneck at the horizon going from right to left. But we can't. Let's think of somebody who wants to make such a trip. If that person starts anywhere on the right and is to end up anywhere on the left, his or her trajectory at some point must have a slope smaller than 45°, that is, he or she will have to exceed the speed of light.

In fact the diagram of figure 14 is somewhat misleading. It shows everything *at an instant of time*. But there simply isn't really time to go from the upper part to the lower part. We cannot go from the real exterior of the black hole to the left quadrant (shown in the lower part of fig 14). Therefore the Einstein-Rosen bridge isn't really a bridge.

One way to think about it is that the bottleneck opens up and closes again before anything can pass through it. But the best way is just to look at figure 13 and say : yeah if we could exceed the speed of light and move horizontally yes we would move from right to left going through the neck at the center. However we are not allowed to do that. We can only move along 45° lines or steeper.

This travelling through wormholes has been the source of all sorts of science fiction, passing from one universe to another through the horizon of a black hole (not into its inside but into its left region). However as you can observe it cannot happen. These wormholes don't allow you to access other universes. They are of a kind called "non traversable wormholes", meaning to say you can't traverse them.

This may come as a disappointment that we won't be able to go on a field trip to other universes at the end of the lectures. But in fact, as we shall see in the next section, there is no real meaning to the left hand side of figure 13 anyway. It is not a real place.

Let's talk now about creating a black hole, not in the laboratory because that is too hard, but in an infinite space by having some in-falling matter.

Formation of a black hole and Newton's shell theorem

In order to study the formation of a black hole, we are going to take a very special kind of in-falling matter. .

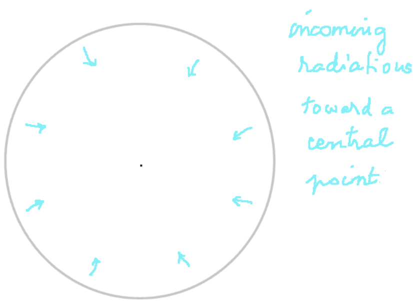

Figure 15 : Shell of incoming radiations.

We begin with a point in space-time, where is no black hole nor anything, and with a very distant shell, figure 15. The shell is not made of iron or other matter like that. It is a thin shell of incoming radiations.

Radiation carries energy. Radiation carries momentum. For one reason or another, it has been created far away. And it is incoming, with sphrerical symmetry, at the speed of light, toward a central point.

At some point in time for the observer, as the shell gets close enough together, there will be so much energy in a small region that a black hole will form. This is the simplest version of black hole formation.

A star isn't really made up out of things which fall in with the speed of light. What we are looking at is the problem of a *black hole* being formed by stuff coming in with the speed of light, the stuff being on a thin spherical shell. It is the simplest of all of relativity problems.

There are only two important things we need to know to understand this formation of a black hole :

1. The shell moves in with the speed of light.
2. We need to know Newton's theorem in classical mechanics, and a version of it in relativity.

Here is Newton's theorem in classical mechanics. If we have a shell of matter forming a 2-sphere, that is, uniformly spread on the surface of a sphere, then

a) The gravitational field *inside* the sphere is null.
b) The gravitational field *outside* the sphere is identical to that of a point mass at the center with the same

amount the mass.

That is Newton's theorem or a special case of Newton's theorem : in the interior of a distribution of mass forming a spherical shell you see no gravitational field, and outside you see the gravitational field as it would be if all of the mass was concentrated at the central point.

That is even true if the shell is moving, for example collapsing toward the center. Of course, as time goes, there will be less inside space, with no gravitational field, and more outside space with a gravitational field. But it is still true.

This theorem is also true in general relativity. It now says the following : if you have a shell of in-falling mass or energy of any kind, then the interior region is just flat space-time. It is like space-time where there is no point source, or no mass. So for the interior, it is similar to classical mechanics. But for the exterior region, it doesn't look like a Newtonian point source with an ordinary gravitational metric around it – because there is no such thing in general relativity. What does it look like then ? It looks like the Schwarzschild metric, that is, the solution to Einstein field equations that we shall study in chapter 9.

So in general relativity, the *inside* of a shell of mass or energy (that is the same thing) is flat space-time. And the *outside* of it looks like a Schwarzschild black hole.

If you have a static, non-moving shell, what you would do to construct the actual solution, that is, the actual metric, is sort of paste together the metric in the interior, which is flat, with the metric in the exterior, which is the Schwarzschild metric.

Notice that we do the same thing in Newtonian physics : we paste together no gravitational field on the inside, with, on the outside, the standard gravitational field due to a central point source. And that is the way we solve problems of Newtonian physics involving a 2-sphere of mass [9]. You do exactly the same thing in general relativity.

Let's redraw the Penrose diagram of a black hole to illustrate what we are going to find, figure 16.

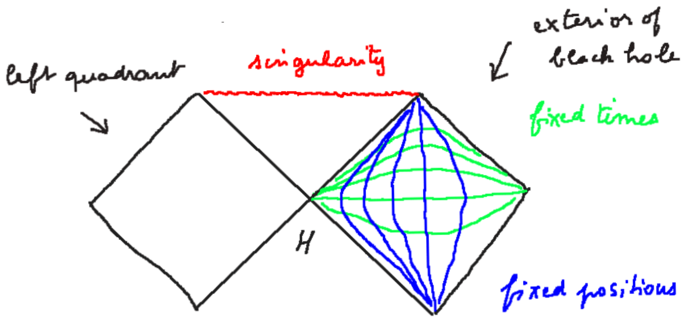

Figure 16 : Penrose diagram of a black hole.

Let's also redraw the Penrose diagram of flat space-time. But we now add an incoming shell of radiations, figure 17. We can think of the incoming shell of radiations as a sort

9. Notice a similar fact in electromagnetism : consider a conducting body with electric charges in a stable, that is static, configuration. Then – whatever the shape of the body – the charges are on the outer surface, and there is no electric field inside the body.

of pulse of incoming photons, a pulse distributed nicely on a sphere.

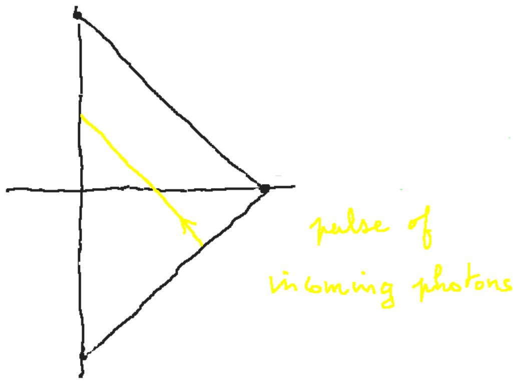

Figure 17 : Pulse of incoming photons in flat space.

The pulse of incoming photons comes from past light-like infinity (what we also call *scry-*). In 3D they form a continuous collection of shells focussing on the origin, but in figure 17 where we can draw only one spatial dimension they are represented simply as a straight line. Nonetheless, let's think of it as a shell.

At any given instant of time, where is the interior and where is the exterior of the shell ? The diagram makes it easy to answer. The triangle represents space-time with R and T. But both shrunk so that the entire half plane of figure 2 fits into the triangle. A point in time and space (being represented with only one dimension) is a point in the triangle, figure 18. The trajectory of the shell is the yellow line. All the events at a given time is the blue curve. At a given time the shell is at the intersection. The interior of the shell is the part of the blue curve to the right. The exterior goes from the shell to space-like infinity.

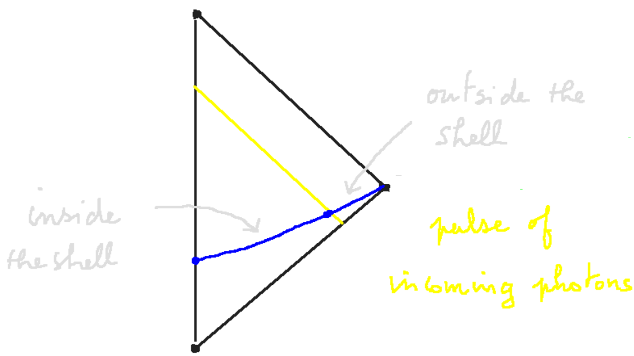

Figure 18 : Interior and and exterior of the shell at any given instant of time.

What Newton's theorem (or its general relativity version) says, is that on the interior of a shell everything is flat space-time.

On the diagram of figure 18, we have a dynamic view of the interior of the shell has it moves. In other words, it is correctly represented by the space-time that we drew in figure 19, which was just a representation of flat space-time.

So, first of all, for the interior of the shell the Penrose diagram of flat space-time is the correct representation of everything that is going on.

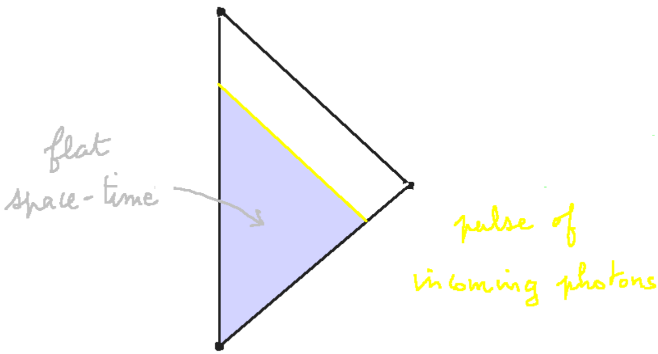

Figure 19 : Flat space-time inside the shell,
as the shell moves in time.

Now what about the exterior ? On the exterior we are told that there is a gravitational field. It does not look like flat space-time. So the upper white region of the triangle in figure 19 is not the correct representation of the physics or of the geometry of the in-falling shell.

Out beyond the shell, what is the correct representation ? It is the representation of the Schwarzschild black hole of figure 16. Let's redraw it without the unecessary details of fixed times and fixed positions, figure 20.

Somebody on the outside (the right square) throws in the shell. The in-falling shell, represented with one spatial dimension, is a radially incoming light-ray. It must move along a 45° straight line. It comes in from far away. It experiences nothing special when it crosses the line called horizon. It keeps going and eventually hits the singularity.

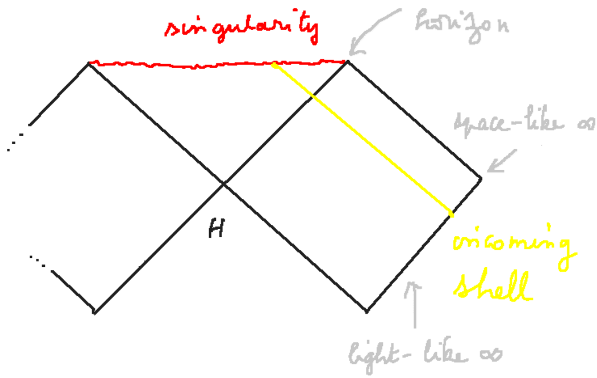

Figure 20 : Shell in the black hole diagram, as it moves in time.

That is what the light-ray would look like on the Schwarzschild geometry.

But now which part of the diagram, in figure 20, is correctly representing the physics that we are doing ? Notice that the interior of the shell is not correctly represented by the black hole diagram of figure 20, because its correct representation is the flat space-time diagram of figure 19. The exterior of the shell in the black hole diagram is shown in figure 21.

So we have two diagrams, one representing correctly the inside and incorrectly the outside, and the other representing correctly the outside but incorrectly the inside.

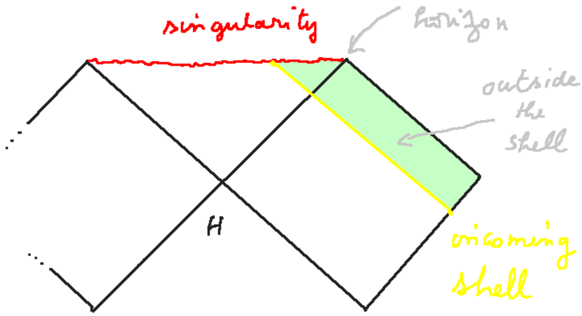

Figure 21 : Outside the shell, over time,
in the black hole diagram.

How do we put the two together in order to make a single geometry ? It is pretty easy. In each diagram we throw away the wrong part, then we paste together the correct parts, figure 22.

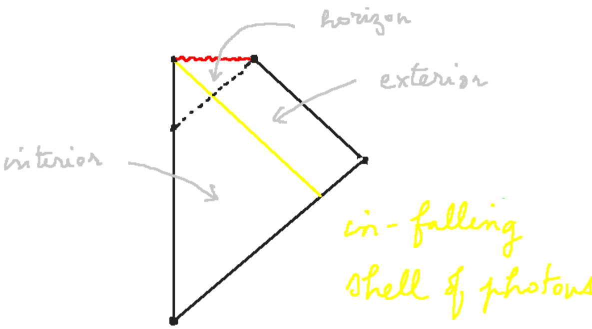

Figure 22 : Interior and exterior of the shell pasted together.

The in-falling shell is the thing that the two parts of figure 22 have in common. On one side of the shell we put flat-space geometry. On the other side we put black hole geometry. The complete diagram represents the geometry of a black hole that is formed by an in-falling shell.

In figure 22, we also plotted the horizon. It is the 45° dotted line ending up at the intersection of the singularity and scry+. Nothing special is experienced by someone crossing the horizon. But once you are inside the horizon, that is, above the dotted line in the diagram, there is no way you can avoid eventually hitting the singularity. Therefore once inside you are doomed. On the other hand if you are outside the shell and outside the horizon, you can get away. You can escape the scry+ for example.

Notice something very curious. Even in the little triangle shown in figure 23, where it seems that life should be trouble-free, you are doomed.

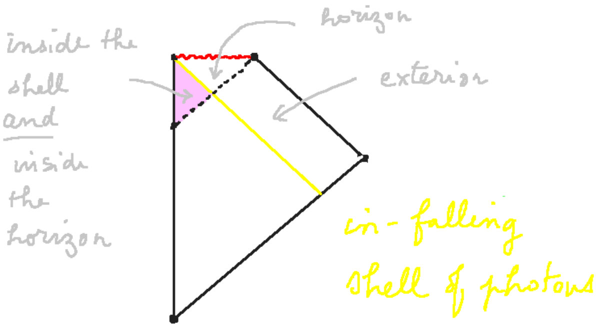

Figure 23 : Inside the shell and inside the horizon.

In the little triangle you are still in flat space-time. The shell hasn't even gotten in yet. You cannot see the shell. If you looked backward on your light cone, you would not see it, because for you looking backward means looking either at light-rays coming from the past parallel to the shell trajectory or on the other 45° line of rays coming from the other direction. In both cases, they don't meet in the past.

So you don't know the shell is coming. You are standing in the flat, apparently trouble-free, space-time region. Yet you are also standing behind the horizon defined by the shell. In other words, even though the shell is still far away, and the horizon still very small, you are already in the doomed region.

To understand this paradoxical situation, let's see what could happen. If you choose not to move in space, the shell will come in and pass you. Then there is no question that you will be inside the Schwarzschild radius. If you choose to move, trying to get out, you can only move on trajectories steeper than 45°. So you will hit the singularity anyway [10], and in a finite amount of your proper time at that.

The horizon itself starts very small at some time in space-time (see next section for a detailed discussion). Then, as the shell approaches, the horizon grows following the dotted line. At some point in time, the shell crosses its horizon. Then the horizon does not grow anymore. In figure 23, suprisingly enough, the dotted line, after the intersection with the yellow line, corresponds to a fixed $R = 2MG$.

10. Whether before the shell caught on you or after, you will hit the singularity.

The usefulness of the Penrose diagram, combining the geometries inside and outside the shell, is to show us that – provided we cannot exceed the speed of light – there is no path out from beyond the horizon, no matter where the shell is.

Of course, if we are inside the shell but outside the horizon, we are not doomed. As long as the shell has not passed us, we can stay where we are effortlessly. Once the shell has passed us, to stay still we will have to fight gravitation with acceleration (like you and me do when we are sitting on a chair – even though we tend to forget it). If we don't fight acceleration, we will fall beyond the horizon and be anihilated when we hit the singularity. But if we accelerate we can not only stay fixed, but we can even fly off to scry+ if we move fast enough.

Let's look at the black hole formation with the more conventional concrete diagram, figure 24.

Inside the shell, space-time is flat. Outside the shell, space-time has the same metric as that of a black hole of the same mass at the center of the picture. The horizon is growing toward the Schwarzschild radius, then it doesn't grow anymore. Whatever is inside the horizon is doomed, *even if the shell is not there yet.*

So long as the shell has not crossed its horizon, it could presumably change its mind and accelerate back out. Well we can suppose that shells of photons don't have minds. But if they had one the photons could turn around as long as they are farther away than $2MG$.

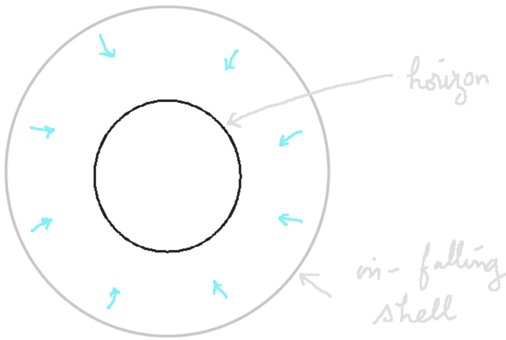

Figure 24 : Shell, black hole and its horizon.

When the shell crosses the horizon nothing special happens, except that now the photons are trapped for good. They can no longer turn around. Or, well, they could try to turn around, but would not get very far. They are now doomed to hit the singularity.

In figure 23, we can visualize the point where the shell (yellow trajectory) crosses its own horizon (dotted line). It happens at $R = 2MG$. Aftewards, we can say that the black hole has formed for good.

Discussion of the paradoxical aspect of the horizon before the shell has crossed it, figure 24 : it seems that as long as the shell is outside its horizon, as in figure 24, someone inside the horizon could possibly move outside of it, while still inside the shell, because in that region space-time is tranquil. Therefore the person would escape from his or her doomed destiny. But that is a mistaken view. There would not be time. Or, if you prefer, the person would have to exceed the speed of light.

That is the superiority of the diagram of figure 23 over the more conventional diagram of figure 24. The Penrose diagram is more abstract, but it shows explicitely the possible and impossible space-time trajectories.

So the horizon is a funny thing. In a ghostly manner, it begin to exists, at first very small, at some point in time after the shell has begun to fall in and is still far away. It grows up to the Schwarzschild radius and then does not grow anymore. The horizon is the point of no return. It is its definition.

Exercise 1 : Suppose there is a spherical mirror at some distance farther than the Schwarzschild radius, and that when the in-falling radiations hit the mirror they are reflected back outward.

1. **Draw in figure 23 the trajectory of the shell of radiations.**
2. **Discuss in which region people are doomed.**

Discussion of the time variable

Let's look more closely at the relationship between figure 23 and figure 24, and at the significance of the time variable.

Figure 24 is the conventional picture of the in-falling radiations. *It is a snapshot at an instant of time.*

Figure 23 is the Penrose diagram for the formation of a black hole. *It shows in the same picture space and time, that is, the entire space-time.*

In order to speak of an instant of time, we have to specify which time do we use ?

Time is just a coordinate. We can make coordinate transformations. To talk about an instant of time really means picking a surface, from a collection of surfaces which are everywhere space-like, see figure 25.

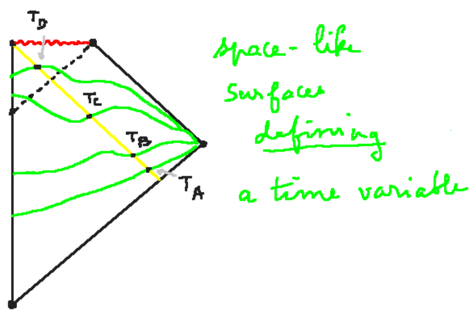

Figure 25 : Defining a time variable.

From this collection of space-like surfaces indexed by some number, time can be taken to be the index of the surface we are on.

If we follow this time variable, at time T_A there is a shell far away. There is no horizon. At time T_B the shell is closer. The horizon has not begun to form yet. At time T_C, the horizon exists, but the radiations are still outside. Sometime between T_C and T_D, the shell crosses its horizon. Then we can say that the black hole exists.

To summarize : as the shell falls in, at some point in time the horizon appears, at first as a tiny point. It grows until the shell meets it, at $R = 2MG$. Then it stays at the Schwarzschild radius. As long as the horizon is below the shell, it is in the flat space-time region. Someone crossing the horizon doesn't feel anything. On either side the person is still in gravitation-free space. However beyond the horizon, even in the flat region, the person is doomed. The boundary is the dotted line of figure 25. It is not when we cross this boundary, that is, the horizon, that we begin to feel gravity. It is when we cross the shell.

The importance of the diagrams cannot be overemphasized. If you try to do calculations to find out where for example the horizon forms, you will find that can't do it. You will wind up drawing the picture, drawing the dotted line from the intersection of scry+ and the singularity, at 45°.

If you want to get a good picture of what is going on, the way to do it is to become familiar with all the diagrams in the lesson, particularly with the Penrose diagrams – for flat space-time, for black hole, and their combination.

If you try to reason the problems out, without a good picture in front of you that accurately represents the relationships between the different parts of space-time, you will run into difficulties.

By the way, the important relationship is not the relative size of things in the pictures. A small circle at the intersection of scry+ and the singularity represents a huge space-time region. The same small circle where the horizon is born, along the vertical side in fig 25, represents a tiny region. The important relationships shown by the Penrose diagrams are what light-rays – which are 45° lines – and other trajectories can link or not.

The diagrams illustrate *causal relationships*, what can cause what, what can signals propagate, who can send a signal to whom. Can a signal from Alice here get to Bob there ? In technical language we would say that the diagrams reflect the causal structure, cause and effect. It means : what can influence what. The rule is that an event E – that is, a point in the space-time diagram – can only influence things in front of it, in other words things in the region where light emitted from E or slower travelling objects sent from E can get to. E cannot influence events outside its light-cone.

The Penrose diagrams were built first of all to be able to draw and examine everything on one page, and second of all to reflect faithfully the causal relations, what can have an effect on what. For that reason these diagrams are very valuable. It is very difficult to think without them. With the Penrose diagrams we see things much more clearly than without them. For instance, we saw how the Penrose diagram of fig 23 is much more talkative than the more conventional diagram of fig 24.

If you play with the Penrose diagrams, it won't take you long to become familiar with them and to be able to use them efficiently.

You may say that the Penrose diagrams are abstract pictures. Yet general relativity is still classical physics. It is much less abstract than for instance quantum mechanics. General relativity can be represented. It is more or less easy of course, but in the end usual intuition works. Not so in quantum mechanics, which, we saw in volume 2 of the collection *The Theoretical Minimum*, is much farther away from ordinary experience.

In the next chapter we shall finally arrive at Einstein's field equations, which are the keystone of general relatitivity. And Lev Landau [11] said of general relativity that it is the most beautiful physical theory ever built.

11. Lev Landau (1908-1968), Russian theoretical physicist. He coined the expression "The Theoretical Minimum" to speak of what one should know to start doing physics.