OUR PHILOSOPHY

OF MATHEMATICS

This booklet presents the five prefaces we have written for our middle and high school books.

Reading them will give you in just a few minutes an overview of our philosophy of mathematics, and how in our opinion it should be taught.

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Preface to the 6th and 7th grade book

We always write the book we would have liked to have in our hands when we were learning the subject it presents.

The two kinds of mathematics books we have chosen *not* to follow are :

1. Books that present a lengthy, logically impeccable sequence of definitions, axioms, lemmas, theorems, and corollaries – sometimes interspersed with examples – in a dense, arid layout resembling a forest of signs, followed by five or six pages of exercises at the end of each chapter.
2. Popular science books that offer a tour of interesting subjects, drawing attention to pleasant, picturesque or quaint mathematical facts, like a walk through a flower garden ; while enjoyable to read, we learn little from them of real substance.

It was in books of the first kind that, as teenager in the 1960s, we learned mathematics during the height of the "modern math" trend in secondary education. These books adopted principles of exposition inspired by a group of professional mathematicians who set out in the 1930s to reframe and present the foundations of the discipline with the utmost rigor.

It was a legitimate research program for advanced mathematics – even if one might argue that it fizzled out – but it was wholly misplaced in secondary education, alienating millions of students worldwide from mathematics.

Since the late 20th century, secondary school textbooks have improved, although traces of this spirit of staunch abstraction and rigor still linger.

However, just as you can't stop a large ship sailing at full speed within a hundred meters – she'll take nautical miles – it will take a generation or two to return to a sensible, engaging, and effective approach to teaching mathematics, suited to students aged 11 to 18.

We hope that our book will contribute to this.

What characterizes it ?

For the writing, we have chosen a conversational style that is neither too informal nor too stiff. We introduce figures from time to time because "a picture is worth a thousand words" and to break up the text. We also introduce very simple literal formulas because they allow us to summarize lengthy sentences. But they are read in English. The fact that these formulas also lend themselves to mechanical manipulation, as Al-Khwarizmi has shown, is a bonus that we do not use in this book.

As for the content, we do not hesitate to explain extremely simple things as well as those that require a little more concentration. Mathematics, in fact, should not be presented only as a vast collection of puzzles or enigmas.

Our goal is not to bestow a certificate of intelligence to those who have been able to read everything, understand everything, and solve all the difficult exercises. Moreover, unlike other authors, we do not indicate the level of difficulty (always subjective) of the exercises or even of the lessons.

On the other hand, we hope to convey what Galileo meant when he said :

"Mathematics is the alphabet with which God wrote the Universe."

Our senses give us access to a set of raw perceptions of nature. It turns out that this set lends itself to description, organization, and interpretation that is largely mathematical. The most profound thinkers have emphasized the astonishing and marvelous nature of this fact.

This organization of our perceptions takes place in our mind. It is so strong that it becomes indistinguishable from external reality itself – like someone blindfolded, gropingly reconstructing what is around them. [1] For centuries, the relationship between the mind (consciousness) and external reality has been a source of questioning for philosophers and scientists. The alphabet that Galileo speaks of is, in fact, more in our heads than in the Universe. We are not going to develop this theme here. Let us simply remember that mathematics, along with many other disciplines, participates in this description of the world around us.

Nor is our goal to teach a set of notions and their rules of manipulation, declaring : "this is how it should be done!" Our aim is more ambitious. We want to begin to help sixth and seventh graders understand what abstraction entails in describing nature, and how it nevertheless always arises from concrete perceptions and intuitive ideas.

Mathematics is essential for engineers, physicists, and practitioners in many other fields. It is also, like foreign languages, history, or biology, a discipline that broadens the mind and helps one feel at ease in understanding the world, even if one has no intention to use math later. We believe that mathematics is part of the foundational knowledge that every student in the general curriculum should have.

We wrote this book for middle school students in 6th and 7th grade, for their parents, and also for students in more advanced grades.

Parents, if they read this book, will be able to refresh – and perhaps even see in a new light – the math they learned long ago, thereby supporting their children. Nothing motivates a child like knowing that their parents are interested in what they are learning, provided the parents do not stress the child or deprive them of the pleasure of discovery, nor write the essays or solve the math problems for them!

1. We might also think of a person who lives solely in the subway, moving from stations to stations. They would construct in their mind a spatial representation of the network very different from that of someone living above ground.

The material presented in this book is generally considered straightforward. It offers an opportunity to reflect on the art of learning. One of the keys to mastering seemingly difficult topics is to proceed slowly, *step by step*. Conversely, it's easy to become confused and to get lost in the simplest concepts if you rush through them "because they are simple". One of the secrets of those who are dazzlingly good at math is that they learned each concept gradually, step by step, like a mountain climber securing every meter of their ascent. When they were toddlers, they spent more time observing each object, each block they played with. And as they grew, they retained this habit. Paradoxically, brilliant people often maintain a simple, down-to-earth, pedestrian approach.

In short, if you want to be good at math, take the time to learn slowly, step by step, what is presented in this book. Don't rush through it just because it is simple.

Students at the end of middle school and in high school will also benefit from revisiting what we explain in 6th and 7th grade.

This book is not only meant to be read ; it is also meant to be reread.

Several people, through their encouragement, information, questions, or comments, prodded me to better express what I wanted to say. First of all, there are my children, for whom I wrote an initial version of the text. There is also Patrick Baronti – of Tuscan descent, who pointed out that the Romans, despite the deficiencies of their numbering system, built the aqueduct passing through the Pont du Gard with a decrease in height of 24 cm per km – Marie Boillot, Michel Lacroix, Marie-Anne Lobjois, and a few others who will recognize themselves. Let them all be thanked here.

Preface to the 8th and 9th grade book

This second volume completes our presentation of middle school mathematics.[2] In both volumes, our goal is to show how simple and natural mathematics can be, even if, as in any discipline, some effort is occasionally required to understand concepts and to handle the tools they present.

We have aimed to show that *mathematics is a language for describing the world and acting upon it.* We have endeavored to eliminate all pedantry and any topic whose sole distinction lies in being a difficult puzzle to solve. The book is primarily intended to be useful by presenting all the mathematics needed in everyday life.

However, it is not merely a how-to manual. Historical sections are included. For instance, we learn how the Babylonians were already solving trinomial equations in the 2nd millennium BC, or how Descartes arrived at the concept of analytic geometry.

We show how the concepts introduced were developed and why the tools presented are presented as they are. We believe this makes mathematics more interesting, easier to learn, easier to understand, and easier to use.

It is also intended to prepare the ground for more advanced mathematical studies in high school and higher education for readers considering careers where they may use a little or a lot of mathematics.

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2. Note on the division of the middle school and high school curricula : We wanted to create a volume for the 6th and 7th grades, a volume for 8th and 9th grades, and then a separate volume for each grade from 10 to 12. For this reason, the 9th grade, which is often considered the first grade of high school, is included here as the final grade of middle school, as is sometimes the case too.

Several principles have guided us :

- We do not hesitate to introduce an intuitive notion before presenting it formally (with definition, properties, and usage).

  For example, one cannot do mathematics without using approximation. The theory was developed by Cauchy at the beginning of the 19th century, and we naturally do not present it here. However, the initial ideas are very intuitive, and we use them informally from time to time.
- Generally speaking, a new theory is easier to understand if one has already encountered examples before studying the theory explicitly. The theory later serves a unifying role.
- Mathematics has a reputation for being a difficult, even esoteric, discipline. *We strongly oppose this view.*

  The view results from the way mathematics has been presented over 2000 years by some mathematicians, though not all.

  Even Euclid provides an absurdly complex proof of Thales' theorem, when a glance at the figure below reveals that, in a plane, triangle ABC can be constructed using nine identical small triangles. Consequently, the segment GH is parallel to BC and has a length equal to two-thirds of BC.

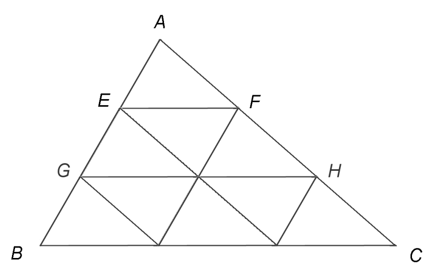

Figure 1 : Proof of Thales' theorem.

Some well-intentioned but ill-advised readers wanting "to show the beauty of math to their 12-year-old child" have asked us how one can prove "using Euclid's axioms" that the two pieces shown in figure 2 can be abutted to form a single straight piece.

Figure 2 : In a plane, these two pieces can be joined to form a straight piece.

We explained to them that, in line with the philosophy of our books, this is a fundamental fact of the plane that requires no proof. We refrained from adding that they risk doing considerable harm to their child by suggesting that "this can be proved using Euclid's axioms". [3]

We are sometimes criticized for not providing the solution to every exercise we propose. But this choice is consistent with our philosophy :

- We do not wish to create only passive students, who solve prepared exercises and simply verify that they are correct.
- We aim to cultivate creativity in students. We encourage them to investigate independently, develop self-confidence, and discover and understand concepts without constant guidance.

3. Euclid's axiomatic approach may hold some interest at the high school level, as an introduction to the concept of an axiomatic framework in mathematics. However, it should then also be explained that Euclid's system is imperfect and was only placed on firm footing by Hilbert in the 19th century. And in our opinion, even that is not very useful before graduate studies.

Note that most of our exercises (as well as much of the text) are very simple. We avoid technical difficulties. Too often, books filled with difficult notions and solutions to difficult problems prevent students from grasping the big picture.

These two volumes provide, on the one hand, a comprehensive presentation of the mathematical tools necessary for everyday life : numbers, fractions, the simplest geometric figures in the plane and space, elementary trigonometry, and a bit of probability and statistics. They cover not only the properties of these tools and how to use them, but also their origins. All of this is presented as simply as possible – without fuss, without pedantry, without kabbalistic notations, and without challenging the reader – aiming only to be didactic and useful.

They also serve as preparation for more advanced mathematics in high school and higher education for those who will continue their mathematical studies. In this second volume, we present functions, curves, basic equations, some results from number theory for cultural interest, geometry that extends slightly beyond the plane and ordinary space, the modeling of random experiments, probability, and more.

The first version of volumes 1 and 2 was prepared fifteen years ago for two of my children when they were in 10th grade. Their subsequent accomplishments, including doctorates for both and careers with major tech companies, would satisfy even the most demanding parent. My two eldest also attained successes I'm proud of, though in fields outside the sciences.

We hope that this book will interest you and that you will find as much pleasure in reading it as we did in writing it.

Preface to the 10th grade book

The 10th grade math course is an important step in our study of mathematics.

In primary school, we learn mathematics that is essential for everyday life : the four basic operations, some elementary geometry, simple shapes, and calculating areas. We also learn to reason logically.

In middle school[4], we revisit the topics from primary school, but taking a higher view and also going deeper. For instance, in arithmetic, we explore the structure of the multiplication table. We learn that $(n+1)(n-1) = n^2 - 1$, which explains why $8 \times 8 = 64$ and $9 \times 7 = 63$. We cover prime numbers, the decomposition of numbers into prime factors, and some fundamentals of set theory – concepts not directly useful in everyday life, but that belong to the general education of a middle school student.

We begin to learn the algebra introduced by Al-Khwarizmi in the 9th century AD, and improved by Viète in the 16th century.

Geometry becomes more advanced : we learn Thales' theorem, widely used in practical applications, reference frames for locating points and drawing figures, the Pythagorean theorem, and elementary trigonometry. Though the Pythagorean theorem is rarely used outside of mathematics, it is a crown jewel of Euclidean geometry. Trigonometry, while rarely encountered in daily life, finds application in certain trades and in navigation.

4. Depending on the school system, the 9th grade is either the last year of middle school or the first year of high school. In our collection of books, the 9th grade is the final year of middle school.

All these subjects lay the groundwork for high school mathematics. Finally middle school students learn to develop organizational skills and acquire a first level of autonomy in their studies.

Now, entering high school, we are moving away from the mathematics needed in everyday life and we begin to study the mathematical toolbox that will be used by the engineer, the physicist, the chemist, the architect, but also by the economist, the financier, the statistician, and even increasingly by the biologist, the sociologist and other practitioners.

One might think that at least in English writing, mathematics is not used. However, artificial intelligence, which is capable of producing texts better written than those of many people, makes great use of mathematics.

In summary, in 10th grade, we step into the more advanced part of the mathematics of the secondary curriculum. Mastery at this level is recognized by the high school diploma, after which some students will pursue the study of higher mathematics in college or specialized technical schools and institutes.

After three introductory lessons on number theory, as developed by Fermat, primarily for cultural enrichment, the book introduces the first major tools in the toolkit of future engineers, physicists, or mathematicians :

- Functions and their graphs. It is important to note that a function in mathematics, as in the phrase “is a function of”, carries exactly the same meaning as in everyday language.
- The simplest equations. While mathematics is often perceived as being primarily about solving equations, this is not our viewpoint. For us, mathematics is a means to describe the world and interact with it. The key step for scientists using mathematics is the *mathematization* of their problem. The questions posed often

become equations to solve, but solving equations is not the essence of mathematics – it is merely a set of techniques. Similarly, knowing how to saw a wooden board does not make one a master cabinetmaker.

- Vectors. They originated as a notational convenience and were gradually adopted throughout the 19th century.
- The connections between algebra and geometry.
- A first non-trivial application of trigonometry.
- Probability and statistics. The book concludes with six lessons on probability and statistics that are more advanced than what we learned in middle school. The ideas and results are presented primarily through simulations.

Without delving into them yet, we lay the groundwork for two fundamental tools : differential calculus and integral calculus. These will be studied in detail in the junior and senior years of high school. Already now, though, we occasionally employ reasoning based on a sequence of increasingly accurate approximations. For example, using this technique, we calculate the area under the curve $y = x^2$ between 0 and 1.

Our course is designed to be accessible to everyone. The exercises are generally straightforward, intended to simply check that the concepts are understood. We are not only addressing the top students in the class ; our goal is to show that mathematics can be interesting and approachable for everyone. We also aim to spark curiosity for more advanced topics.

Greater emphasis is placed on grasping ideas than on the completeness or flawless rigor of proofs. It is well known that one can memorize and reproduce a proof without truly understanding it or seeing its practical value.

In lessons on probability and statistics, for example, we present few formal proofs. Instead, we offer numerous simulations to help develop an intuitive understanding of chance.

Considering that 10th grade mathematics is part of the general knowledge expected of a well-rounded person in the 21st century – alongside subjects like English, history, geography, one or two foreign languages, biology, and other fields – we freely refer to these disciplines. We also draw connections to philosophy, particularly when it touched upon science in the pre-Socratic era.

We believe that school learning is often too passive. At best, it produces learned minds of a rigid variety, but rarely fosters creative and conceptual thinkers. [5] Our aim is to move beyond producing merely learned minds. We strongly encourage self-study and investigation beside the curriculum.

Learning how to learn is equally important, though, in our view, it remains insufficiently emphasized in traditional education. Admittedly there has been progress during the 20th century – particularly in contrast to the 19th century, when textbooks often consisted of rote questions and answers to memorize.

Throughout our teaching, we stress the importance of going *step by step*. It is the author's opinion that going step by step, and constantly relying on the right models in the mind [6], there is no discipline that can't be learned.

When studying a new subject or tackling a challenging problem, it's important to recognize the value of taking breaks – ideally every two hours. Step away, do something else for a while, go walk the dog. Avoid working when you're tired, as this often results in poor work and harms one's self-confidence. For difficult problems, it's helpful to review them

5. Learned minds of the rigid variety belong to category II in the classification proposed by Gaston Bachelard (1884, 1962) in his remarkable work, *The Formation of the Scientific Mind*, Clinamen Press, 2006, originally published in French in 1938. Category I includes those who accumulate knowledge without organization, believing that television games test true knowledge. Creative and conceptual minds, found in category III, are the ones education should strive to cultivate.

6. It is an idea we develop in the preface to our 11th grade book, taking up the luminous remark of Michel Talagrand (born 1952), Abel Prize 2024 : "It's not technical obstacles that block. It's when we look at the problem from the wrong point of view."

before bed, even if you haven't solved them, and allow your brain to work on them while you sleep. Often, the solution will be in your mind, shining clearly, when you wake up. This is one of the remarkable abilities of the human brain ; it is far more complex and powerful than we typically realize. [7]

We would like to express our gratitude to Jean-François and Joëlle Roux, as well as Michel Lacroix, for their stimulating mathematical conversations during the writing of this book. We also extend our thanks to Denis Blanchet for his consistently insightful suggestions to improve the text.

As always, it was a great pleasure for us to present the mathematics of the 10th grade in this book. We hope that you will get just as much pleasure from reading it and from learning or revising mathematics at this level.

7. We catch a glimpse of this when we recall our dreams, though that remembrance is often fleeting. The brain can be like those teachers who erase the board too quickly, before we've had the chance to write everything down :-)

Fortunately, when it comes to solutions for math problems – or problems in general – they usually don't vanish like dreams.

Preface to the 11th grade book

"It's not technical obstacles that block. It's when we look at the problem from the wrong point of view."

Michel Talagrand
Abel Prize, 2024

To approach mathematics with ease, we need the right models in our head. This is essentially what Michel Talagrand says in his own way. These models are mainly elementary and geometric in nature, and we build them in childhood.

When we encounter new mathematical concepts, it can sometimes take time to find the right models to understand and work effectively with the new concepts. Until we succeed, "we don't understand". The new concepts seem like mere mental games involving obscure and unintelligible ideas. We think we're bad at math. Notice that the same is true in many other disciplines, such as psychology, sociology, history, economics, money (especially) [8], biology, medicine, and more.

When teaching or writing books, it is important to understand why students or readers do not understand, or do not understand well, certain ideas. We are in a favorable position when we have been there ourselves, when, at the beginning, we did not understand certain notions, when we did not have the right models in our heads, perhaps because the notions were difficult, or perhaps especially because they had not been presented to us clearly or even correctly. The first

8. In the following paper we explain the wrong model most people have of money : `https://lapasserelle.com/billets/greek_crisis.html`

time I was told about negative numbers, when I was 12, I was told that they were *very strange* creatures. And for a long time, I pictured them as a kind of translucent numbers or inside-out gloves. Later, I realized that they were simply, and only, marks on the line of numbers to the left of zero, obeying simple rules (addition is a shift to the right, subtraction to the left).

Humans and other mammals constructed a Euclidean representation of space in their minds eons ago. It works very well. Its only flaw is that it makes it a little more difficult later to understand non-Euclidean geometries.

For example, if we are on a warped rubber surface, it is difficult for the human mind to understand that we could get our bearings and work on this surface without necessarily plunging it into a three-dimensional Euclidean space to visualize it. We could imagine ourselves as a little ladybug, unaware of dimensions other than the two in which it moves, taking measurements of length and angles of all kinds to figure out the geometry of its space. This is even more true when there are three dimensions around us, but the space is not Euclidean. [9]

Along with a good understanding of the elementary geometry of space, another prerequisite is to have a good grasp of the fine structure of the line of numbers. It is important to understand the positioning of integers, fractions, and real numbers that are not fractions. After a little effort, we realize that fractions do not fill the entire line, even though they are dense within the larger set of real numbers.

We are then in a good position to understand that the concept of approximation in mathematics plays two distinct roles. On the one hand, approximations allow us to obtain

9. Just as on Earth, if we walk straight for a long time, we return to our starting point by arriving from the other side, there are comparable geometric spaces with more dimensions. This may be the case for our universe. We know that it is vast, but it may be finite like the surface of the Earth. It would only be locally Euclidean. There is much more to say on the subject, but this is not the place in this preface.

rough values of solutions to problems, which are often sufficient. We frequently did this in middle school. But they also have a more fundamental role : they allow us to *define* many objects as the limit of a sequence of objects. This is the important idea with which we begin the book.

Euclid did not use approximations. However, they enable us to demonstrate certain results much more simply than he did, such as Thales' theorem.

Equipped with these tools – a good understanding of the elementary geometry of space and an understanding of the topology of the line of real numbers – we can understand mathematics with ease up to the first years of college. After that, it remains accessible, because we have become accustomed to representing new concepts with models that we already have in our minds.

There are pitfalls to avoid. The first is to start with the idea that a new concept is complicated. Most mathematical concepts, if not all, are actually simple when we have the right models to represent them. This is true of vectors, limits, derivatives, integrals, elementary combinatorics, complex numbers, abstract spaces, stochastic calculus, etc. I intentionally chose examples traditionally considered intricate by those who fall into this first pitfall.

The second pitfall is to try to go too fast. You must always proceed step by step when studying new concepts. If you need to build software with thirty-six similar objects in complex interaction, start by building the same software with just two objects.

We mentioned this in the 10th grade course, but let's repeat it : great mathematicians and great physicists are not exceptionally intelligent people, but they are people who have *very clear ideas* and, I would add, who have simple, effective, functional models in their minds. These are models whose basic elements they built when they were toddlers. These models are never very complicated, even if they can sometimes seem far from the usual representation we have of the space in which we live.

It is true – I'll grant you – that seeing something familiar in a new light requires creativity and a flexible mind.

The junior year course is a transitional course between the sophomore and senior years. The sophomore year was a significant step compared to middle school, and the junior year is often seen as a less significant step. But that is not our view. There is still a long way to go to reach the final year of high school. We have included in this book results that some might consider better suited for 12th grade (such as major theorems of projective geometry), but which seem easy to understand, if not to demonstrate.

When we were learning elementary analysis at the turn of the 1970s, textbooks were stuffed to the brim with epsilon-delta arguments (what we call "epsilonitis") because for many years teachers and authors considered that this was real mathematics. Thankfully, this is no longer the current approach. We have included a little epsilonitis to show how to demonstrate certain convergence results using the topology of the line of real numbers. However, most of the time, we have relied on geometric intuition.

In mathematics, as in other disciplines, we hold onto our first impression of a subject and its difficulty for a very long time. This is why it is especially important, as a teacher, never to introduce a new concept by saying that it is complex.

As students, we should always approach a new domain by telling ourselves that it is simple. It may take time to build the right model in our minds – this is the most important step – and to learn all the details, but ultimately it will be simple. In other words, we should start with the belief that what we are going to learn is not complicated, because, in general, it is simply true.

We try to remain interesting. We invite people to think, but *without stress*, to discover new things or perspectives that are not necessarily difficult. On the contrary, we often progress more slowly than many teachers. We know that the most delicate points will be understood quickly by some students and a little later by others. That's of no import.

When something was explained to Hilbert too quickly, he was notorious for not understanding what was being said. Yet everyone agrees that he was the greatest mathematician of the turn of the 20th century.

We do not wish to dumb down either. You will not find in our 11th grade book problems like the following : starting from an integer between 0 and 10, we added 7 a certain number of times and arrived at 61. What was the starting number ? And how many times did we add 7 ? We found this, in the chapter on sequences and series, in an 11th grade cramming manual.

This is the kind of exercise that allows you to consistently get A's in high school, but it does not prepare you for the tough selection process to gain admission to the best colleges, or the effort needed to do real math at university.

We avoid like the plague formulas that are only formally very complicated (kabbalistic), requiring intense reflection only to reveal obvious facts. Denoting the projective plane with an expression like

$$\mathbb{R}P^2 = \{\mathbb{R}^3 \backslash (0,0,0)\} / \sim$$

even if it can be justified, seems to us preposterous. The notation P^* serves the purpose just as well.

One last recommendation : the other day we were watching a subject on TV where several students from a top school were typing text, eyes glued to the keyboard, using just two or three fingers. So let's repeat a comment we already made : learn to type on a keyboard with all ten fingers.

It's not just a matter of efficiency ; it's a matter of ethics.

That a person of a certain age does not know how to type is understandable. But young people of 16, born in the computer age, no.

When you do something, do it well.

My friends and regular interlocutors have once again accompanied me through lively conversations during the writing of this book. May they be thanked. A special thank you to my brother, who made the remark on epsilonitis.

Happy reading and good learning of 11th grade mathematics.

Preface to the 12th grade book

The twelfth grade mathematics course is the keystone of the mathematics curriculum we have been studying throughout middle and high school.

In middle school, we learn the mathematics necessary for everyday life, along with some additional material for general knowledge. In high school, we begin to learn the mathematics necessary in engineering, physics, and other fields, but it remains at an elementary level.

This book is also an introduction to higher mathematics that some of the readers will study in college. The course covers a wide variety of subjects and is organized in four parts :

I Analysis : integral calculus and series of functions

II Algebra : complex numbers, algebraic structures, linear algebra and matrices

III Probability and statistics

IV Programming

In analysis, it deals with elementary integration which completes differentiation presented in our 11th grade math book.[10] We have not covered multivariate analysis, the beautiful theorems of Green-Ostrogradski or Stokes.

In algebra, we deal with complex numbers which we have been talking about for several years. We stay on the surface of their theory. It is very important and very useful for the engineer, but their complete treatment is beyond the high school level.

After introducing general algebraic structures, we cover the fundamentals of linear algebra, including endomorphisms,

10. *High school mathematics : 11th grade*, Eagle's Beak Press, fall 2024.

vector bases, matrices, and matrix operations. We explain changes of basis but mention only briefly diagonalization and do not talk about spectral theory.

Between the second and third parts, we chose to present two topics – recurrent sequences and elementary combinatorics – that show elegant and non-trivial properties, which can be grasped without abstraction.

The third part is devoted to probability and statistics, going beyond previous years' material. We present Gaussian distributions and the Central Limit Theorem. In statistics, after reviewing estimation and hypothesis testing, we dedicate a lesson to Bayesian techniques, which are experiencing a revival in applications like search engines, spam filtering, large language models (e.g., ChatGPT and its competitors), and other uses.

In each of these parts, the topics covered are only an introduction to wide domains.

The book ends with a part on algorithms and programming, primarily explained through examples. We have chosen not to recount in detail the wonderful history of computing, which is as exciting as that of the industrial revolution. Doing so would have required four or five lessons and would have unbalanced the course. After briefly presenting algorithms, we provide some general ideas of what programming entails by writing a small program in HTML/JavaScript. The simulations illustrating the operation and applications of Bayes' theorem were also written in HTML/JavaScript.

From primary school to the final year of high school, the exposition of mathematics mixes concrete models and abstractions : a plane is the top of a table, or the contact surface obtained by rubbing two stones against each other "in all directions". It is also a two-dimensional vector space. [11]

Some mathematicians smirk at this approach, saying that it is a naive presentation of the discipline. They are the same

11. Or an affine space linked to a two-dimensional vector space.

who did so much harm to the teaching of mathematics by introducing set theory and “modern math” in middle school.

The greatest mathematicians – those who have made the most important contributions to the discipline – do not have such an attitude. Jacques Hadamard (1865, 1963) in his mathematics course for high school does not hesitate to say that a plane is a two-dimensional surface characterized by the fact that if two points are on it then any straight line that passes through them is also entirely in it. And for a straight line, he says that a taut thread provides an image of it. [12]

In the final year of high school, we continue to blend concrete models and abstractions.

The only drawback of this approach using concrete models is that sometimes, without us realizing it, our models incorporate unnecessary axioms. That’s why it was so difficult to develop non-Euclidean geometry, and it took mathematicians more than two thousand years to do it (from Euclid to Lobachevsky (1792, 1856)).

We frequently refer to physics because, in our view, a mathematics course that ignores physics is not a good mathematics course. At best, it is a theoretical corpus of abstract results, cf. the sharp critique by Vladimir Arnold (1937, 2010) of mathematics education disconnected from reality. [13]

In our lessons, exercises are incorporated into the main text to encourage the reader to immediately apply what has just been explained. To achieve complete mastery of the topics covered, the reader will also need to consult other textbooks that offer more exercises than we do.

We seek to give a high-level vision of mathematics free from any apprehension, but also one that truly understands it.

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12. J. Hadamard, *Leçons de géométrie élémentaire*, Armand Colin, 1898, available here : `https://lapasserelle.com/documents/geometrie_hadamard.pdf`

13. Available here : `https://www.math.fsu.edu/~wxm/Arnold.htm` or here `https://www.lapasserelle.com/arnold/arnold-eng.html`

Many people understand (or think they understand) a problem at an emotional level. Sometimes they are even able to talk about it for a while. But for various reasons, they are unable to produce anything useful. One of the reasons is that they use mental models of the math concepts they try to understand, that are not correct. Therefore the models cannot lead to a deep understanding.

We talked about this in the preface to our 11th grade math book, where we quoted the luminous remark by Michel Talagrand (born 1952), 2024 Abel prize :

"It's not the technical obstacles that hold us back. It's when we look at the problem from the wrong perspective."

Our goal is to help students build effective operational mental models. For most people, as it is for the author, these models are of geometric nature. We go into some details – it is important to go into details to master a domain, moreover it allows us to check that our mental models work –, but we avoid drowning in them. In short, we aim to train people who will be able to

– understand mathematics,

– use tools and apply methods,

– and even create.

This preface is not the place to discuss the usefulness of general education for all. The concern arose at the end of the 19th century, and it is clear that the objective has not been fulfilled. The problems of the school, which we can learn about daily in the press, have diverse and deep causes. Let us simply observe that the general public often has dismaying ideas, and ultimately has a very poor knowledge of science. The latter has indisputably become more complex in the 21st century than in the past.

A characteristic of the technological innovations introduced in the 20th century is that most of them were understandable to the general public. They were at least in their uses (an automobile, a telephone) if not in the principles on which they were based (thermodynamics, electromagnetism). Those introduced in the 21st century, by contrast, are less easy : genetic engineering, the prowess of computer science.

Popular vision of computers with sheaves of 0 and 1.

A documentary film on computers will inescapably include a few sequences showing sheaves of 0s and 1s forming beautiful figures on the screen against a dark blue background evoking mysterious processes – which, let's agree, explains nothing. In this, science, after having been for a century, from 1880 to 1980 approximately, one of the vectors of public education, is once again becoming an arcane knowledge in the hands of Merlin the Enchanter.

At the same time, we are witnessing in television documentaries, whether touristic, geographic or ethnographic, a return of the supernatural and the magical. The voice-over presents with the same objectivity the methods that the filmed tribes use to weave baskets or to conciliate spirits.[14] The phenomenon can be observed in society at large : in the 21st century one makes more money selling clairvoyance than selling knowledge.

We have a certain way of understanding mathematics that guided our exposition in the middle and high school volumes. This way of understanding mathematics is closely linked to the understanding of the world, to the representation of nature, to physics, to epistemology and one can even say to philosophy.

14. Pick any TV documentary, and count how many seconds it takes before the voice-over talks about spirits. Usually, it is less than 30 seconds.

Our method consists of staying as close as possible to a concrete understanding of the world. [15] You will find people who do not understand that in a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. But you will find no one who, wanting to go from one corner of a rectangular field measuring 90 m by 120 m to the point catty-corner, doesn't understand this observation : if you go around the field, you will walk 210 meters, but if you cut across the field, you will walk only 150 meters. The same idea guides all our explanations. We venture into abstraction only when necessary.

For us, mathematics is by no means a separate, exceptional, platonic discipline, inaccessible to ordinary mortals. Mathematics is simply the well-structured, often quantitative part of the way we organize perceptions in our brain.

We believe that this approach leads to a better understanding of mathematics. There are difficulties in mathematics, that is clear. But there are also difficulties in understanding the world whatever the domain, be it biology, history, psychology, sociology or anything else. When we study mathematics simply as a set of tools to understand the world, it is easier.

If you understand our books from sixth grade to twelfth grade, you will be well-prepared to follow later math courses in college and beyond.

We hope you enjoy reading this book as much as we enjoyed writing it, and we wish you good luck for the future.

15. This doesn't mean we dislike abstraction. It is quite the opposite. We even think that reality *is an abstraction* that was built over hundreds of millions of years in the brains of mammals. But that is not a good way to teach math in high school :-)

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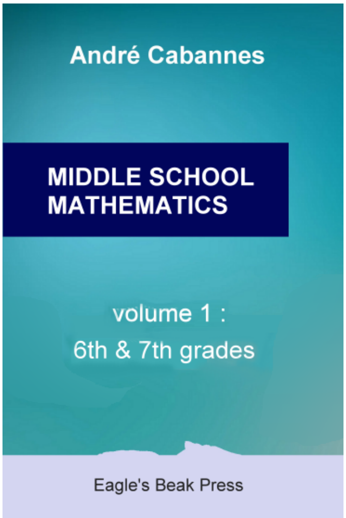

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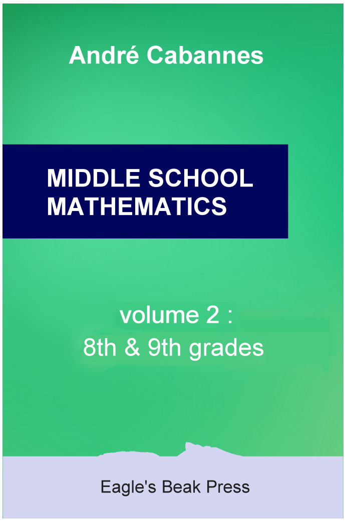

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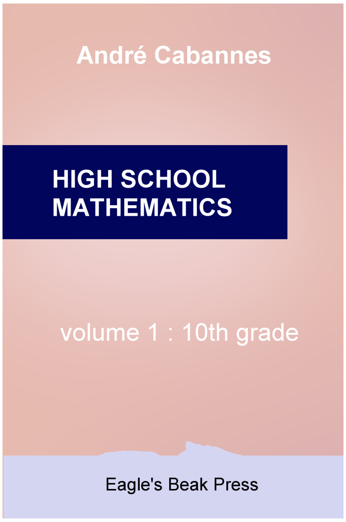

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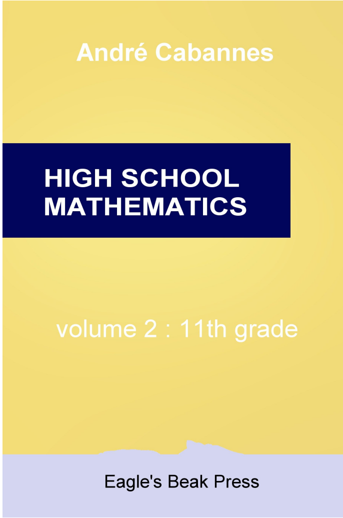

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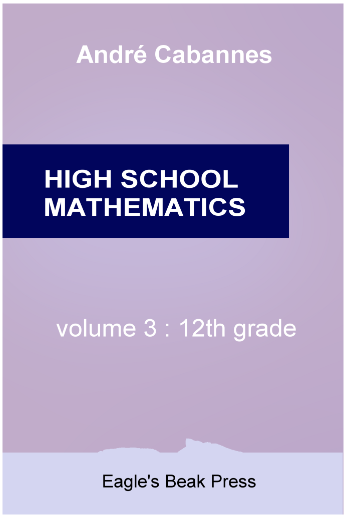

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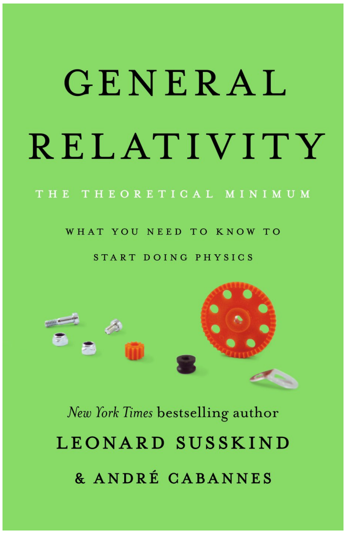

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